THE GOLDEN AGE OF INDIAN MATHEMATICS

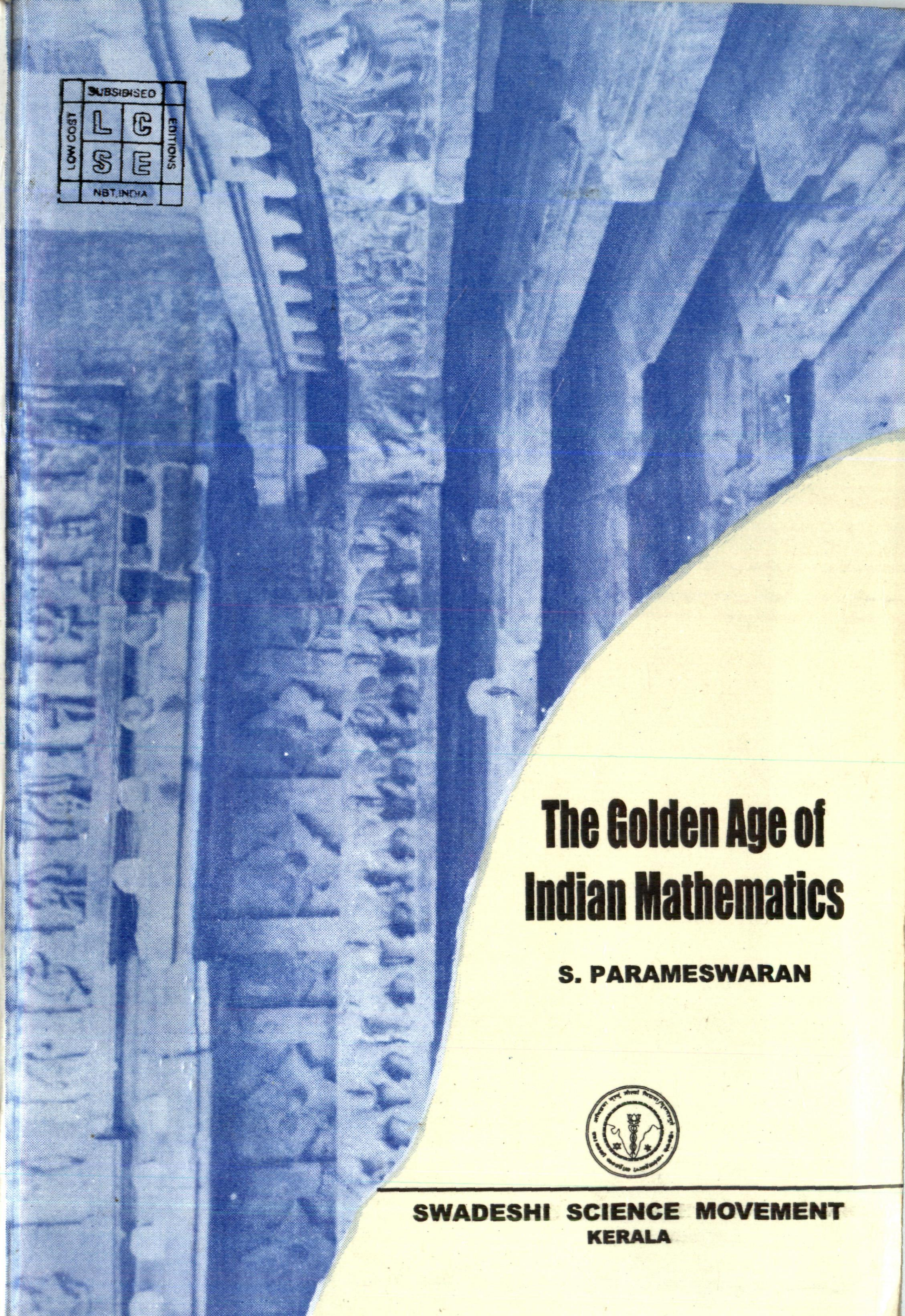
Till recently it was considered that Indian mathematics remained barren during the period between the thirteenth century and the eighteenth century. Now it has come to light that it embalms the "Golden Age" of Indian mathematics. The earlier mathematicians were working on mathematics of the finite - trigonometry, mensuration, algebra etc. Kerala, the southern most part of India produced a mathematician by name Madhavan (c. 1340 – c. 1425), who broke the finite barrier and soared to the infinite. The infinite series expansions he reaped anticipated the discoveries of mathematicians like James Gregory (1638 – 1675), Newton (1642 – 1727) and Leibnitz (1646 – 1716). In short, Madhavan deserves to be called the father of infinitesimal analysis.

Dr. S. Parameswaran, a renowned professor of Mathematics, was born on September 1920 in North Parur, Kerala. He passed his B.Sc (Honours) degree in 1941 in first class, first rank and joined the University College at Trivandrum the same year as Lecturer in Mathematics. He secured an assistanceship for research in the University of Illinois (USA) in 1957 and he was awarded Ph.D degree by that University in1960. Apart from a number of Mathematics Text Books he wrote for college students, he has to his credit about 200 technical and popular science articles published in periodicals and more than a dozen popular science books. He had served as founder-president of Kerala Mathematics Association and National Forum of Mathematics Education. In recognition of his merits as a teacher he was appointed on retirement from service as Emeritus Professor of Mathematics by the University Grants Commission, India.

Price Rs. 42/-

The Golden Age of Indian Mathematics

SUBSIDISED LOW COST EDITIONS LCSE NBT.INDIA



The Golden Age of Indian Mathematics

S. PARAMESWARAN

SWADESHI SCIENCE MOVEMENT
KERALA

The Golden Age of Indian Mathematics

S. PARAMESWARAN

Published by
SWADESHI SCIENCE MOVEMENT
KERALA

THE GOLDEN AGE OF INDIAN MATHEMATICS

Dr. S. PARAMESWARAN

First Published 1998

Rs. 42.00

SUBSIDISED
LOW COST
L C
S E
EDITIONS
NBT.INDIA

This book has been subsidised by the Government of India through the National Book Trust, India for the benefit of students.

Printed and published by Dr. M. N. Sreedharan Nair on behalf of Publications Division, Swadeshi Science Movement, Convent Road, Kochi-35, Kerala, India.
Printed at Akshara Offset, Ph: 471174, 473470, Tvpm.

PUBLISHER'S NOTE

THE GOLDEN AGE OF INDIAN MATHEMATICS is the second in a series of books the Swadeshi Science Movement proposes to publish on the scientific heritage of our country. The Golden Age refers to an extremely fruitful period in the history of Indian mathematics when remarkable contributions were made to the advancement of certain branches of Mathematics by the mathematical genius of Kerala, the southernmost part of India. This era was preceded by the period extending from mid-seventh century to mid-fourteenth century AD which saw the advent of commentators like Haridatta, Govindaswami, Sankaranarayana, Suryadeva, Govinda Bhattathiri and others who brought out commentaries on the great works of Aryabhata and Bhaskara I (both considered to be from Kerala) and introduced refinements and improvements on their methods of calculations. The Golden Age is the glorious period from 1350 AD – 1600 AD, marked by original contributions of Madhavan, Parameswaran Namputiri, Nilakantha Somayaji, Jyeshta Devan, Sankara Variyar, Puthumana Somayaji and Sankara Varma.

We are indeed grateful to Dr. S. Parameswaran for the excellent work he has done in bringing to light and interpreting in the language of modern mathematics the rich and varied contributions of Kerala's great mathematicians which, for centuries, remained in

the darkness of the 'Grandha Puras' (store houses of manuscripts and books) of Palaces and Namputiri 'Illams' (houses) until they were collected and exhibited in 1832 by Mr. C. M. Whish, the East India Company Civil Servant.

We are also indebted to the National Book Trust, India for their financial assistance in the publication of this work.

Publisher

PREFACE

An ideal title for this book would be "The Golden Age of Keralaese Mathematics", as the mathematical discoveries during the "Golden Age" *were entirely due to mathematicians of Kerala. Yet in a broader perspective* these were essentially Indian contributions to the world of Mathematics. Hence the title "Golden Age of Indian Mathematics."

Indian Mathematics may conjure up the names of Aryabhata, Brahmagupta, Mahavira and Bhaskara of olden days. The discoveries of these mathematicians fall under two categories: one in which they were anticipated and the other in which they anticipated others.

The position of Aryabhatiya of Aryabhata in India is akin to that of the Elements of Euclid some eight centuries before. A comparison of the achievements of Indian mathematicians with those of mathematicians of other countries is given, after the preface.

Till recently it was considered that Indian mathematics remained barren during the period between the thirteenth century and the eighteenth century. Now it has come to light that it embalms the "golden age" of Indian mathemtics. The earlier mathematicians were working on mathematics of the finite - trigonometry, mensuration, algebra etc. Kerala, the southern most part of India produced a mathematician by name Madhavan (c. 1340 -

c. 1425), who broke the finite barrier and soared to the infinite. The infinite series expansions he reaped anticipated the discoveries of mathematicians like James Gregory (1638 – 1675), Newton (1642 – 1727) and Leibnitz (1646 – 1716). In short Madhavan deserves to be called the father of infinitesimal analysis.

This book is in two parts : the historical and the mathematical. The first part contains a survey of the history of the land, the biographical details of the mathematicians and just a mention of their mathematical contributions. This part may be smooth sailing for the general reader. The second part contains demonstrations of the mathematical results mentioned in the first part. The demonstrations have been taken from the original sources and presented in as faithful a manner as possible, so as to enable the reader to get an idea of how the minds of those mathematicians worked. A knowledge of junior college mathematics will be sufficient for understanding the material in the second part.

A glossary of terms is also appended.

February, 1998 **Dr. S. PARAMESWARAN**

ACKNOWLEDGEMENTS

A project on " Keralese contribution to mathematics" that I undertook as a University Grants Commission awardee was the starting point of this book. I thank the University Grants Commission, India for the award.

I am indebted to many people for the various types of help received but specially to Professors K.V. Sarma (Madras), V.S. Sharma (Trivandrum), P.U. Krishna Variyar (Kottayam) as well as S. Hariharan (Bangalore) and Ullur P. Ramanathan (Trivandrum) for supplying me with material needed for the book. But for the valuable advice and generous encouragement rendered by Professor Dirk J. Struik (M.I.T., U.S.A.) this book would not have attained its present form. My indebtedness to him is beyond words. My gratitude goes to Professor Jeremy Gray (Open University, England) for his helpful suggestions. Last but not the least I thank my wife Ponnammal for her forbearance and patience, our sons Kumar and Kishore as well as Mrs. Susan Kumar for their efficient help and co-operation in the preparation and handling of the manuscript.

My thanks also due to Swadeshi Science Movement, Kerala for taking up the responsibility of publishing this book with the support of National Book Trust of India.

Some achievements of Indians, upto the end of the 12th century A.D., are compared with those of other countrymen. The reader can get an idea of the anticipations made on either side. This is not the period we study in this book. It is intended to set the stage for our discussion.

1. **Pythagoras theorem.**

The Sulva sutras (800 - 500 B.C.) of India, contain the statement "The diagonal of a rectangle gives an area equal to the sum of the areas given by its length and breadth". [102].

* * *

The Chinese work K'iu - ch' ang Suan - shu or Arithmetic in Nine Sections, written before 1000 B.C. states : "Square the first side and the second side and add them together; then the square root is the hypotenuse". [100, I & II]

Pythagoras (572 - 501 B.C.) [102]

2. **Pythagorean triads**

The Sulva Sutras (800 - 500 B.C.) of India include a statement about Pythagorean numbers i.e., numbers satisfying the relation

$x^2 + y^2 = z^2$ [100, 1]

The Sulva Sutra of Apasthamba gives the rules for constructing right angles by stretching cords of the following lengths: 3, 4, 5; 5, 12, 13; 8,15, 17; and 12, 35, 37. [7]

Brahmagupta (India) in his Brahma Sphuta Siddhanta (628 A,D) gave the two sets:

$2mn,\ m^2 - n^2,\ m^2 + n^2$

and $\sqrt{m}$, 1/2 (m/n - n), 1/2 (m/n + n)

[100, I]

Bhaskara (India; 1150) adds two further relations

m, $2mn / (n^2 - 1)$, $m (n^2 + 1) / (n^2 - 1)$ and

$m (n^2 - 1) / (n^2 + 1)$, $2mn / (n^2 + 1)$, m.

[100, I]

* * *

The tablet from the old Babylonian period (c. 1900 to 1600 B.C.) contains the Pythagorean triple

$p^2 - q^2$, 2pq, and $p^2 + q^2$, where $p > q$. [7]

To Pythagoras has been attributed the rule for Pythagorean triads given by

$(m^2 - 1) / 2$, m, $(m^2 + 1) / 2$, where m is an odd integer. [7]

A formula for Pythagorean triples

2n, $n^2 - 1$, $n^2 + 1$, where n is any natural number, bears the name of Plato (b. 427 B.C.) [7]

3. Pascal's triangle and Combinations

Sushruta Samhita written by the Indian physician Sushruta (6th century B.C.) says that 63 combinations can be made out of the 6 rasas or tastes (bitter, sour, saltish, astringent, sweet, and hot) taking them one, two etc. at a time. (note that we get 6, 15, 20, 15, 6 and 1 combinations, which add up to 63).

Pingala (3rd century B.C.), an Indian writer on prosody, considers in his Chandas - sutra the method of finding the number of combinations obtainable by taking one, two etc. letters out of a given number of letters. The meaning of the rule is difficult to understand. A commentator of the 10th century A.D., by name Halayudha explains the meaning by the construction of a diagram called Meru Prastara,

which is the same as 'Pascal's triangle'.

Mahavira is, perhaps, the world's first mathematician to give the general formula

nCr = [n(n-1)(n-2) ....(n-r+1)] / 1.2.3. .... r

This occurs in verse 218 of his Ganita Sara Sangraha (850 A.D.) [102]

* * *

Ssu-yuan yu-chien (Precious Mirror of the Four Elements) (1303) of Chu Shih-chieh (China) gives the triangular form of numbers, commonly known as 'Pascal's triangle. (The four elements are the heaven, earth, man and matter). [7].

The Arabic mathematician al-Kashi (1436) gives the binomial theorem in 'Pascal's triangle' form. [7].

Pascals's triangle appeared in print in 1527 on the title page of the arithmetic of the German mathematician and astronomer Petrus Apianus. [7].

Blaise Pascal (1623 - 1662).

4. The approximate value 3. 1416 of pi.

Aryabhata (India) in his Aryabhatiya (499 A.D.) states thus: "add 4 to 100, multiply by 8 and add 62,000; the result is approximately the circumference of a circle whose diameter is 20,000."

* * *

Apollonius (262 - 190 B.C.) of Perga in his Quick Delivery probably gave this value; Ptolemy (150 A.D.) used the value 377/120 in his Almagest. [7].

5. Cube-root of an Integer

Aryabhata (India) in his Aryabhatiya (499 A.D.) gives the rule

for the extraction of the cube root of a number. "As regards cube-roots, we have so far no evidence of the method (given by Aryabhata) having been known earlier." [102]

* * *

Liber Abaci (1202) of Leonardo Fibonacci or Leonardo of Pisa contains a chapter on square-roots and cube-roots. [100, I].

6. **Indeterminate equations of the first degree** (known as linear Diophantine equation)

Aryabhata (India) in his Aryabhatiya (499 A.D.) discusses the method of solving (in integers) equations of the type:

$N = ax + c = by + d$ or $ax - by = k$.

Apparently the Indian mathematician Brahmagupta was the first one to give a general solution of the equation $ax + by = c$, where a, b, and c are integers. This was in 628 A.D. Brahmagupta knew that if a and b are relatively prime, all the integral solutions of the equation are given by $x = p + mb$, $y = q - ma$, where m is an arbitrary integer and $x = p$, $y = q$, a solution of the equation. [7].

* * *

Diophantus (250) of Alexandria had been satisfied to give one particular solution of an indeterminate equation. [7].

7. **Diophantine Quadratic equation $y^2 = 1 + px^2$** (named mistakenly for John Pell (1611 - 1685)

Brahmagupta solved the equations for $p = 83$ and 92.

The Indian mathematician Bhaskara (1114 - 1185) solved the equations for $p = 8, 11, 32, 61$ and 67. For $p = 61$, he gave the solution $x = 226, 153, 980$ and $y = 1, 776, 319, 049$. This is an impressive feat in calculation, and its verification alone will tax the efforts of the reader. [7 ; 102].

* * *

Archimedes (287 - 212 B.C.) of Syracuse proposed the Cattle Problem, which involves the solution of the above equation for p = 4,729,494, incidentally provides a first example of the above equation. [7].

Diophantus (250) gave single answers to the equations for p = 26 and 30. [7].

8. **The identity $x^2 = (x - y)(x + y) + y^2$**

Brahmagupta's Brahma Sphuta Siddhanta (628 A.D.) contains this.

* * *

This result was known to the Greeks and is called the formula of Nicomachus (c. 100 A.D.) of Gerasa, [102].

9. **Area and Diagonals of a cyclic quadrilateral**

The Indians Brahmagupta (628) and Mahavira (850) gave the following formulae for the area and diagonals of a quadrilateral:

Area $= \sqrt{(s-a)(s-b)(s-c)(s-d)}$, where a, b, c, d are the sides and s is the semiperimeter; and diagonals are

$$\sqrt{(ab + cd)(ac + bd) / (ad + bc)} \ \& \ \sqrt{(ac + bd)(ad + bc) / (ab + cd)}$$

The formulae are correct only for the cyclic quadrilateral; both Brahmagupta and Mahavira do not mention this.

* * *

The first of these formulae was rediscovered by W. Snell, who gave it in his edition of Van Ceulen's works. [100, II].

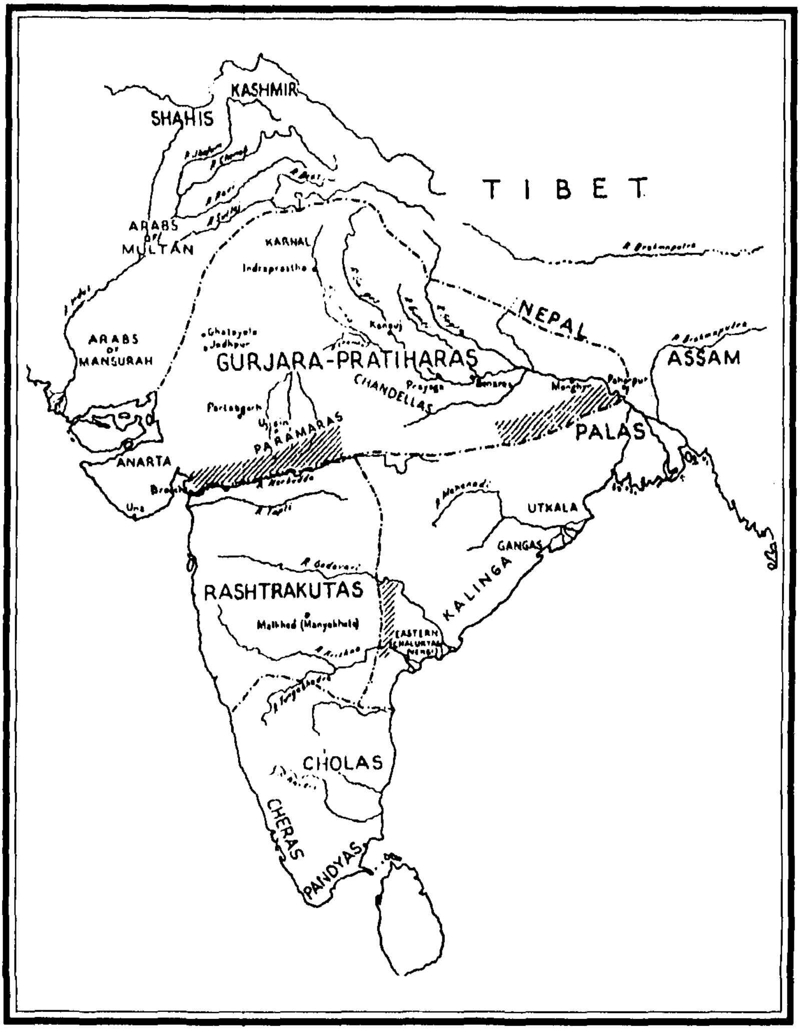
10. **$\sin(A + B) = \sin A \cos B + \cos A \sin B$**

Bhaskara's Siddhanta Siromani (1150 A.D.) contains this result.

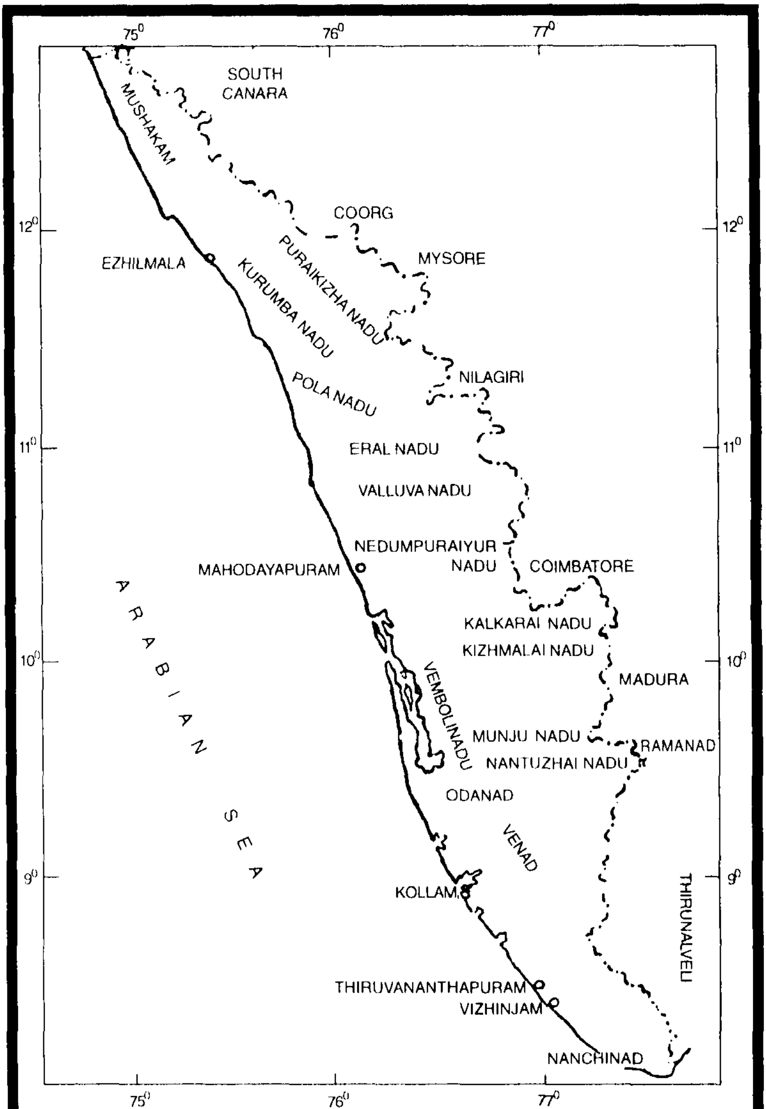
* * *

This formula is named for Ptolemy (100 - 178 BC).

It is generally believed that Ptolemy's Almagest owes much to the work of Hipparchus (180 - 120 B.C.). [3; 7 & 91].

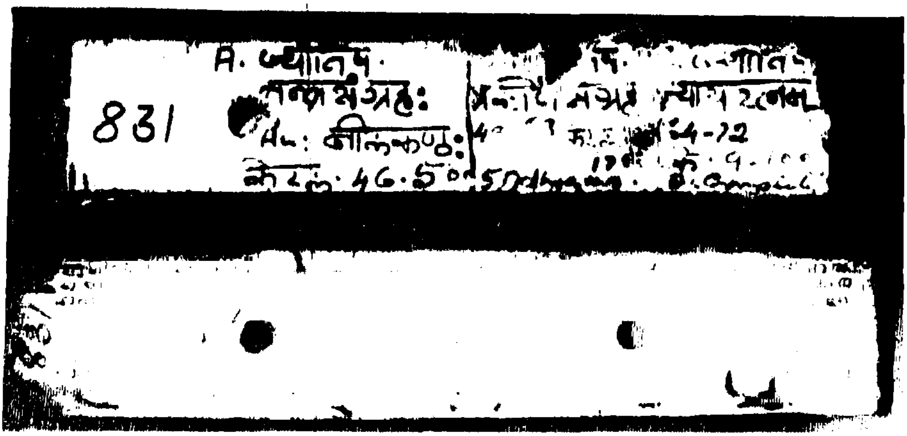


1. Ancient cities and Kingdom of India

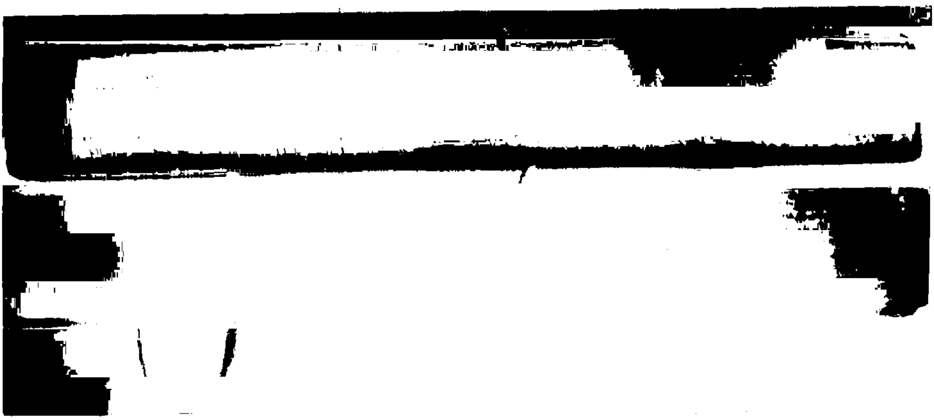


SOUTH CANARA
MUSHAKAM
COORG
MYSORE
EZHILMALA
PURAIKIZHA NADU
KURUMBA NADU
NILAGIRI
POLA NADU
ERAL NADU
VALLUVA NADU
NEDUMPURAIYUR NADU
MAHODAYAPURAM
COIMBATORE
ARABIAN SEA
KALKARAI NADU
KIZHMALAI NADU
MADURA
VEMBOLINADU
MUNJU NADU
RAMANAD
NANTUZHAI NADU
ODANAD
VENAD
KOLLAM
THIRUNALVELI
THIRUVANANTHAPURAM
VIZHINJAM
NANCHINAD

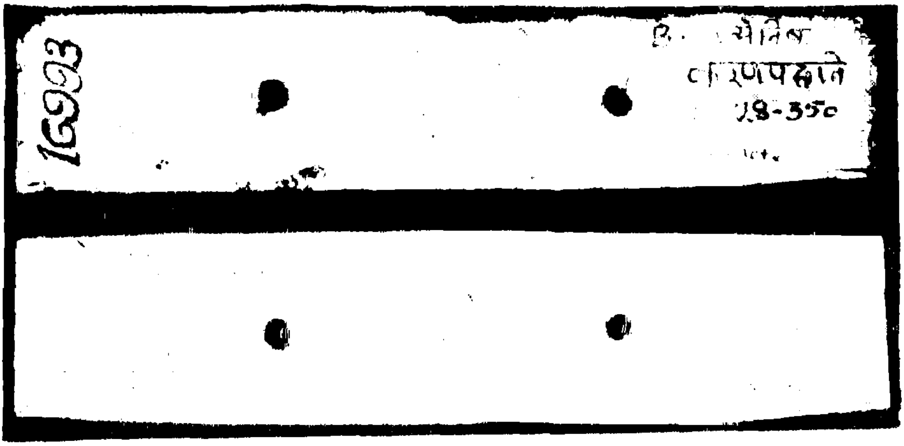
2. Kerala



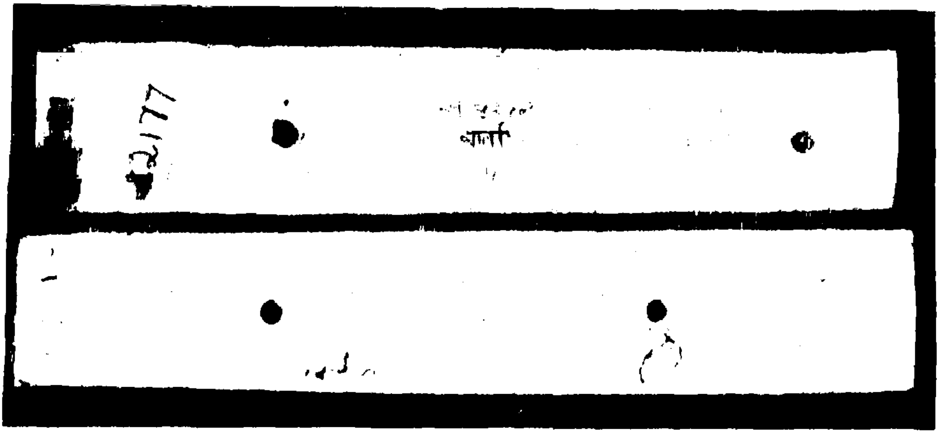
3. The title page (Sanskrit) and the first page (Malayalam) of the palm leaf manuscript of Tantra Samgraha.



4. The title page (Sanskrit) and the first page (Malayalam) of the palm leaf manuscript of Yukti Bhasa.



5. The title page (Sanskrit) and the first page (Malayalam) of the palm leaf manuscript of Karana Paddhati.



6. The title page (Sanskrit) and the first page of the Introduction (Malayalam) of the palm leaf manuscript of Sadratnamala.

CONTENTS

PART - 1 HISTORICAL SECTION

PART - II MATHEMATICAL SECTION

LIST OF ILLUSTRATIONS

PART I

HISTORICAL SECTION

INTRODUCTION

The narrow strip of land, known as Kerala, lying along the south-west coast of the Indian peninsula, starting from near the southern tip, can be described as a land of treasures. Its spices and other gifts of nature attracted the trading community from several European countries, centuries back. The treasure of beauty and charm that nature has lavished on this landscape thrills the aesthetic sense of any visitor and makes him exclaim that this is really a tourists' paradise. These proud possessions of Kerala are well known the world over. Kerala literally means the land of coconuts.

HISTORY OF THE LAND

How the land originated:

We have two versions on the formation of the strip of land. One is based on a legend: more than three millennia ago there lived a brahmarshi (brahmin turned a sage) named Parasurama (pronounced PARASHU - RAH - MA). He is considered an incarnation of Vishnu (a member of the Hindu Trinity). Standing on the southern shore of the Indian peninsula he flung his parasu (a kind of axe) into the ocean; the ocean receded upto where the axe struck. The land came to be called Kerala. A contradictory legend goes as follows: Once upon a time there lived a king named Maha Bali. He ruled over the state of Kerala. The land was at its best during his regime; equality among his subjects, justice for all, peace and prosperity to everyone prevailed. Vamana (pronounced VAH-MANA), an incarnation of Vishnu, appeared then in the form of a brahmin boy. Vamana literally means short-statured. The boy tricked and pushed Maha Bali into the nether world. Maha Bali made a last request: that he be permitted to visit Kerala and see his erstwhile subjects once a year. Vamana granted him his wish. The day of Maha Bali's visit to Kerala is celebrated as Onam. It is a festival, in which all the people of Kerala, irrespective of their caste or creed participate. In the serial order of Vishnu's incarnations,

Vamana is the fifth and Parasurama is the sixth. How can Maha Bali rule a land which Parasurama created later? Chronological sequence of the two legends leaves the reading public confused as to what to believe.

The second version is based on the views held by geologists and historians. Geologists think that some natural disturbance such as an earthquake 'threw up' a piece of the ocean bed; this they say happened long before the time of Parasurama. There is no consensus among researchers on the date of Parasurama; apart from some wild guesses putting it about 6000 B.C., or 4,000 B.C. some assign it to c. 1200 B.C., or a little earlier.

Let us see what the ancient history of south India has to offer. The story of any civilization is beset with numerous hurdles. History combined with legends, facts with fiction, conclusions with surmises make the data hazy and sometimes contradictory. One has to weave a somewhat coherent picture out of this tangle. There will be different versions and view-points, gaps, obscurities and dearth of definite dates etc. We try to present here an outline of the saga of events concerning south India, of which Kerala is a part, from as early a period as possible.

Dravidian civilization and north India

When the curtain of history rises on the Indian peninsula, Dravida, the old name of south India, was inhabited by a stock of people known as Dravidians. It is now generally believed that at some very early period (according to some, about 4000 B.C) Dravidians were in occupation of the greater part of India. Excavations at Mohanjo daro (in Sind) and Harappa (in Punjab) have led archeologists to recognise an ancient but unguessed civilization on the Indus valley. People inhabited the city of Mohanjo daro continuously for a period of atleast six hundred years from 3300 B.C. to 2700 B.C. after which the city was aparently deserted. Excavations have also shown that the city was a product of a long established civilization. As regards Harappa, the tale is different. The people who occupied the

neighbouring villages later, tampered with the old setting and partly ruined the archaeological treasure. The excavations indicate that the city was not occupied after 2500 B.C. The Indus valley cities do not exhibit any ancient palaces or royal tombs. They reflect a high degree of comfort for the ordinary people and thus are the oldest planned cities ever discovered.

It is believed that the type of civilization evidenced at the Indus valley cities was essentially Dravidian in origin.

The language of the Dravidians is also called Dravida. It does not belong to the Aryan or Semitic groups. Dravidian family now includes the four major languages, the Tamil, Telugu, Kannada, Malayalam and many other minor languages and dialects. The Gondi, Kui, and Kurukh are Dravidian languages spoken by primitive people dwelling in isolated 'pockets' in central India. There is an 'island' of Dravidian-speaking people living around Calett city, in Baluchistan, on the far side of the Indus. Their language is known as Brahui. Those people are not known to have any contact with the Dravidian culture within historical times. It is clear that Dravidian influences must have existed in the north-western corner of India in the remote past.

The Aryans who entered India from the north-west waged a fierce war over a long period of time with the Dravidians, whom they found in possession of the land. Gradually they made themselves masters of north India. The Dravidians retreated to the southern portion of the great peninsula. Aryans did not attempt to advance by force of arms beyond the dividing line of the Vindhya hills. Later they came to the south as peace-loving cultural missionaries. The Dravidians welcomed them and held them in great respect and affection. The newcomers mingled with the original inhabitants and thus came a fusion of the two ancient civilizations.

When did the Aryans invade India? There is a belief that the Aryans entered India about 3000 B.C. Another belief is that the inhabitants of Mohanjo daro and Harappa had to flee, leaving the

cities deserted sometime before 2500 B.C. The invasion lasted for a long time. So we can say that the Aryans entered India and occupied north India during the first half of the third millenium B.C. [44, vol. I & 92].

This is the story of the Dravidian civilization that once prevailed in north India. Next we take a bird's eyeview of the panorama of the Dravidian civilization in its homeland viz., south India.

Dravidian civilization and south India

South India presents a curious unsolved archaeological puzzle. Primitive copper, bronze and iron objects have all been found lying mixed together in prehistoric sites but iron objects far outnumber those of copper and bronze. Can it be that, long before and independently of the rest of the world, south India had discovered iron for itself? Numerous iron deposits exist in south India, easily worked from the surface. Remarkable skill in mining and metal work was developed in south India from remote times. A prehistoric gold mine exists at Maski, in Hyderabad state. Remains of primitive pottery and articles of polished stone and metal are most commonly found in megalithic sites associated with prehistoric burials.

The Dravidians were great agriculturists. It seems that their ancestors were the first people in the world to grow paddy plants, which yield the grain paddy from which rice is taken. The very word rice owes it origin to the Tamil word arisi. The existence of numerous terra-cotta spindles in their ancient grave chambers, taken together with the fact that cotton is a native plant of Deccan (as the region south of the Vindhya hills is called) shows that the ancient inhabitants of south India were familiar with spinning and made cotton clothes. They were also expert fishermen. On the south-eastern coast of the great peninsula, archaelogists have found the remains of a very ancient pearl fishing establishment. As far back as history can probe, there seems to have been no time when south India was not in contact with a number of foreign lands by way of sea. The sea-faring people

of the west coast (i.e., the present Kerala) traded with the people of Asia Minor, Egypt etc., this activity was in a flourishing state four thousand years before Christ. Some of the important seaports which continued to maintain the close maritime ties with the middle east were Machiri (the present Kodungallur), Tondi (the presnet Kadalundi) etc. located on the west coast of Kerala. References to this appear in several works of the Sangham period - a period from the second century B.C. to the fourth century A.D. all the evidences indicate the existence of a widespread and stable prehistoric civilization throughout south India.

[44, Vol. I & 92].

Dravidian languages

The original Dravida language is what is known as Pazhan Tamizh (or old Tamil). Later it got divided into two: the northern Dravida and the southern Dravida. The northern Dravida branched off as Kannada and Telugu. The part of the Indian peninsula south of Kannada and Telugu speaking areas was known as Tamilakam. Akam means land; hence Tamilakam is the land of the Tamils. Kings belonging to the three dynasties called Cera, Pandya and Chola held sway over portions of Tamilakam from very early times for a very long period. How early and how long are not known. The western side of Tamilakam was known as Cera, the south-eastern the Pandya and the north-eastern the Chola. The word Kerala derives from Cera and so Kerala is an alternate name for Cera.

Tamil is the purest of the Dravidian languages. This is borne out by the fact that the Tamil in pandya region has the fewest Sanskrit words in its vocabulary. Tamil poetry recited by bards at the courts of the Tamil kings have been transmitted orally for centuries before it was written down. Tamil as a litereary language originated about the third century before Christ. The oldest surviving work on Tamil grammar is Tolkappiyam. Tolkappiyar, its author was a student of a scholar named Agastya, who is believed to have lived in the second

century B.C. (This Agastya is different from the sage Agastya who lived at the time of the Ramayana). The scholar Agastya had composed a Tamil grammar titled Perakattiyam (or great Agastyam). This work has not come to light so far; but some rules from this work are found quoted in the commentaries on Tokappiyam.

When Tamil blossomed into a literary language the scholars in the three kingdoms Cera, Chola and Pandya began to compose their works in that language. A Tamil Sangham or literary academy flourished for about six hundred years from the second century B.C. to the fourth century A.D. Madura, the Pandyan capital was the literary center or headquarters of the Sangham. Many outstanding works of high calibre were produced during this period. It is scarcely possible to give even a bare summary of those works. Among all the Tamil classics, the most popular is the Kural also called Tirukkural. It was composed by Tiruvalluvar, a poet of great genius who lived in the first century A.D. In terse couplets the philosopher-author brings his shrewd experience and pointed wisdom to bear on the ethical values and problems of his day. The twin epics the Cilappadikaram and its sequel the Manimekalai are very famous. The first one was written by Ilankovadikal, a monk - brother of a Cera king in the second century A.D. It has for its theme the love of a wealthy young merchant (Kovalan) for a beautiful dancer (Matavi) and the sorrows and ultimate sacrifice of the faithful wife (Kannaki). The second epic narrates the story of the dancer's daughter, Manimekalai, who after hearing the religious truths expounded by teachers of various schools finally embraces Budhism. This is the work of a Buddhist grain merchant, Caattanaar of the second century A.D.

Among the trio, the Cera kings were the most illustrious. In wealth, might and honor they were supreme. They were great patrons of literature and lavished awards and gifts on the scholars and poets. Some works, like the Patittupattu of the Sangham period eulogise the qualities of Cera kings. [44, Vol. I & 92]

With the advent of Tamil literary works, an artificial language

bound by grammatical rules etc. evolved. This "refined' language was called Cen-Tamil, while the earlier 'unrefined' or layman's language was called Kodum-Tamil. Though Kodum-Tamil prevailed over the three kingdoms, it exhibited variations from region to region. Thus the Kodum-Tamil of Cera (Kerala) differed from of Pandya and Chola. Gradually the Kodum-Tamil of Pandya and Chola regions was replaced by a language which was closer to the scholarly Cen-Tamil. Kerala retained the original south Dravida or Kodum-Tamil language with certain individualities. This language got more and more estranged from the language of the rest of Tamilakam and became an independent language. The consensus among the scholars is that the original south Dravida language is what we now call Malayalam and that Cen-Tamil and its allied language prevalent in Pandya and Chola regions originated later.

The style and diction of the works produced during the Sangham period were beyond the reach of the common man. So the Shaivite Nayanars and Vaishnavite Alwars composed devotional poems in simpler language closer to the common tongue of the people. This happened between the seventh and ninth centuries A.D. The language of these works is also called Cen-Tamil, though it differed very much from the Cen-Tamil of the Sangham period. [44, Vol. I ; & 92].

The political history of south India

The political history of south India is a complicated one. We confine ourselves to a bare outline of the sequence of events, which has a bearing on Kerala. The three dynasties of Cera, Pandya and Chola were very friendly to one another until the end of the tenth century. They fell into subjection under Pallavas, an intruder-gang, who appeared suddenly on the scene at the end of the third century A.D. and disappeared as suddenly in the tenth. With the accession of Rajaraja I, who ruled Chola from 985 to 1014, the great period of Chola expansion was ushered in. He destroyed the fleet of the Ceras. His son Rajendra I during his reign from 1014 to 1044 continued his conquest. Rajendra overthrew the Cera king, Bhaskara Ravi Varma

in 1018 A.D, divided the conquered territory into small states and left them to the care of local chieftains. the decline of the Chola dynasty started during the dawn of the thirteenth century and was complete in 1267 A.D. With the decline of the Chola dynasty many small states in Kerala began to shake themselves free. In 1730 A.D. Martanda Varma ascended the throne of Travancore, the southern-most state of Kerala. He annexed several small states that lay to the north of his state and consolidated them to the state of Travancore. This he achieved by 1750 A.D. The original capital of Travancore was Padmanabhapuram; but from very early times Trivandrum had a special status. The temple of Sri Padmanabha, the family deity of the rulers of Travancore, is located at Trivandrum. The rulers of Travancore had to be present at the various rituals conducted at the temple. So Trivandrum was a second home for the rulers until about 1790 A.D. Thereafter Trivandrum became the capital of Travancore. Padmanabhapuram and its neighboring portion of Travancore now belongs to Tamil-nadu.

The Cera kings were known as Ceraman Perumals. They had their capital at Mahodayapuram or Tiruvancikulam (the present Kodungallur). The last of the Perumals was Bhaskara Ravi Varma. Ancestors of Cochin kings, as descendents of the Perumals, continued to reside at Mahodayapuram. It was through the gradual merger of smaller territories, received as marital gifts from the local chieftains, that Cochin gained in size and status. Until the dawn of the fifteenth century Kodungallur was the capital of Cochin state; later Cochin city became the capital and Trippunithura, the royal township.

During the thirteenth century the Zamorins (the anglicized version of Samootiri i.e., kings) of Kozhikode (Calicut), in the northern part of Kerala, gained more and more power and prosperity; by the fourteenth century they had become a power to be reckoned with.

[44, Vols. I, II, III; 92].

Kerala became a single state in 1957, when the Government of

India reorganised its states on a linguistic basis. Prior to that Kerala consisted of three regions: the southern region was Travancore, the middle one Cochin and the northern one Malabar. Travancore and Cochin were ruled by maharajahs or kings while the Governor of Madras province held sway over Malabar. Cochin state had its own postal system; Travancore state had its own postal system and coinage. In the Malabar area there were a number of rajahs, something akin to local chieftains. The maharajahs and rajahs, in keeping with the tradition of the Cera kings, were great patrons of art and literature; some were sound scholars and/or good artists.

Kerala is one of the thickly populated regions of India. Also it has the highest literacy rate in India; many districts of Kerala have declared complete literacy. In 1957 the communist party came to power by the democratic process of adult franchise and formed the first ministry in the newly carved Kerala State.

* * *

Parasurama is supposed to be the first brahmarshi to leave the land of the Aryans and move southward. People who went with him were the progenitors of a sect of brahmins known as namputiris (pronounced NAM-POO-TIRIS).It is plausible that this migration across the hills of Vindhya (Deccan) happened about twelfth century B.C. or earlier. Namputiris settled down in sixty-four villages in Kerala first and gradually spread out to other parts of Kerala. They became landlords. Among the people who came over to Kerala there were few women. So they decided that only the eldest son in a family should marry from the community ; other sons can marry from the families of rajahs, Nairs etc. The eldest son alone inherited and managed the family estates. This arrangement safeguards the family estates against fragmentation. The younger sons had no responsibility for or say in family matters; the talented among them took to artistic, literary or scientific pursuits. In short, conditions in Kerala were favourable for all kinds of creative activities. [44, Vol. I].

Keralese culture

Kerala has a rich tradition in art, literature and science. We can easily fill volumes expounding them; here we shall rest content with a bird's eyeview of the panorama.

Researchers have unearthed a huge collection of folklore that flourished long time back in Kerala. This collection has helped to revive certain forms of art and culture which had almost vanished. Some of these primitive forms are now considered as precursors of the more sophisticated items of today.

Kerala was the birth place of a form of art called Kathakali. Kathakali literally means enacting a story. It is a type of silent dance drama. The male actors used to appear on the stage in gaudy attire and gorgeous makeup, as both male and female characters. The mode of communication is not by word of mouth but by gestures. Every single gesture or facial gesticulation has a meaning. The gestures symbolise the songs rendered by musicians standing on the stage behind the actors. There is simultaneity of word and action. The monopolistic sway of male artists on the kathakali stage has become a thing of the past. In central Kerala there stands an institution known as Kalamandalam where prolonged and rigorous training is imparted in this traditional art form.

Kathakali combines art with literature; the written piece of narration goes by the name of attakatha (pronounced AH-TTA-KATHA). It has high musical and literary values. Authors of attakatha were usually scholars in both literature and music.

The tradition of music in Kerala can be proud of having a royal composer, Swati Tirunal Rama Varma (1813-1846 A.D.) as its votary. Some connoisseurs of music assign this royal composer a place close to if not on a par with the all time great musical trinity of south India, namely Tyagaraja, Muthuswamil Dikshitar and Syama Sastri. There is an institution called Swati Tirunal college of music which specializes in the teaching of music both vocal and instrumental. This

college is situated at Trivandrum, the capital of Kerala State.

Architecture and mural paintings seen in temples and palaces speak volumes about the achievements of Keralese artists in those domains. The oil paintings of Raja Ravi varma (1848 - 1906 A.D.) have won acclaim from artists both Indian and foreign. Ivory carving took great strides and became a fine art.

Malayalam (pronounced MALA-YAH-LAM) is the regional language of Kerala; it has been the common language of the three regions namely Travancore, Cochin and Malabar from very early times. The early form of this language was the original south Dravida. Under the influence of the Aryans Malayalam developed closer and closer ties with Sanskrit. Malayalam of today has more affinity to Sanskrit than to any other language. It is not difficult for a person versed in Malayalam to follow Sanskrit to some extent. Malayalam is now a well developed language and its literature is rich and extensive. Through translations, adaptations, commentaries etc., Malayalam has brought also into its fold many of the important works in several languages, Indian as well as foreign. In short Malayalam now serves as a window for a view of world literature. The word Malayalam is a fine example of a palindrom. *Malayala - kara* is another name for Kerala. *Mala* means hill, *alam* sea and *kara* land. *Malayala - kara* is the land lying between the hill and sea.

The three main systems of medicine namely, allopathy (or western medicine), ayurveda and homeopathy exist side by side in Kerala but ayurveda, also called the indigenous system, alone has a hoary tradition. Some techniques developed in this system are considered unique; many people from outside come to Kerala to undergo this treatment. Kottakkal, Olassa, Vayaskara are names of great spas preserving the tradition.

Many great savants and preceptors of *Jyotis-sastra* (the science of celestial luminaries) flourished in this tract of Indian peninsula at different periods of time. They produced several

authoritative treatises dealing with theoretical studies and practical applications of celestial investigations.

A picture of Kerala will be incomplete if we do not mention the name of Sankara (pronounced SHANKARA) of the eighth century A.D., the illustrious (advaita or monistic) philosopher that Kerala ever produced. He was the greatest spokesman for the monistic doctrine which tolerated no God apart from the all-inclusive One and taught that Brahman (which is the Pure Being, Pure Knowledge, Pure Bliss) is the sole reality; all else is Maya or illusive appearance.

Scholars of ancient as well as later periods used to compose their works, whatever be the topic, in verse. A good number of those treatises were written in Sanskrit language. Names of places, houses etc., which were in Malayalam, could not be included as such in Sanskrit works. It was customary to sanskritize those names so that they could fit in with the Sanskrit diction.

In this part of the country, a treasure consisting of works on astronomy (including mathematics), medicine, architecture etc. had been lying hidden to the outside world for generations. Only a few fragments of this 'treasure-trove' have so far been ferreted out. Even these bits indicate that this treasure is in no way less charming than the scenic beauty and is perhaps more valuable than the spices and other crops. When compared with similar findings from other parts of the world and belonging to the same period, the state of knowledge indicated by some of these fragments is really astounding.

* * *

From very early times, the people of Kerala held the study of *Jyotis - sastra* (science of celestial luminaries) in high esteem. Naturally it attracted the attention of a good number of people; some turned out to be ardent devotees of that science. *Jyotis-sastra* comprises two parts: a theoretical part and a practical part, the latter usually being referred to as the predictional part, The phases of the moon, solar and lunar eclipses, and variations in the movements of

"Planets" etc. are some of the events which can be calculated in advance. These as well as descriptive details pertaining to the earth and other celestial bodies belong to the theoretical part. Reading of horoscopes forecasting future events of persons, countries etc., reckoning of muhurtams (auspicious moments) etc. fall within the scope of the predictional part. Some people use the terms astronomy and astrology to denote the theoretical part and predictional part respectively. Mathematics was an important and helpful tool to the study of *Jyotis-sastra* and was fostered as such. The result was that those mathematical topics like mensuration, properties of circle, sphere etc. needed in the study of *Jyotis-sastra* claimed the prime attention of enthusiastic votaries of, the muse of astronomy and gained enormous development.

Our principal interest is the mathematical lore of Kerala. Does not an archaeologist proceeding on his excavatorial investigations, stop for a while to enjoy the sight of a beautiful scenery or a colorful garden, on the wayside? Just so we may halt occasionally to gaze at the astronomical wonders appearing in our horizon, while proceeding on our journey along the highway of mathematics.

History of Keralese mathematics resembles a jig saw puzzle with many missing pieces; some parts of the picture are fairly clear; others may be reconstructed with the aid of a controlled imagination but many gaps remain which perhaps may never be filled.

SYSTEMS OF NUMERATION

Let us begin with a discussion of some of the important systems of number - representation that prevailed in India.

Indians had separate names for the powers of ten, up to 17:-

eka = 1, *dasa* = 10, *sata* = 100 - 10^2, *sahasra* = 1000 - 10^3 *ayuta* = *10,000* = 10^4, *niyuta* (also called *laksa)* = 100,000 = 10^5 *prayuta* = 1,000,000 =10^6, *koti* = 10,000,000 = 10^7, *arbuda* = 100,000,000 = 10^8, *abja* (or *vrinda*) = 1,000,000,000 = 10^9, *kharva* = 10,000,000,000 = 10^{10}, *nikharva* = 100,000,000,000 = 10^{11}, *mahapadma* = 1,000,000,000,000 = 10^{12}, *sanku* = 10,000,000,000,000 = 10^{13}, jaladhi =100,000,000,000,000 = 10^{14}, *antya* = 1,000,000,000,000,000 = 10^{15},= *madhya* = 10,000,000,000,000,000 = 10^{16}, and *parardha* = 100,000,000,000,000,000 = 10^{17} [11; 16 & 61]

Even though several systems for expressing numbers sprouted in ancient India, only a few gained any popularity and acceptability.

The *Bhuta Samkhya* system and the *Katapayadi* system were two schemes which survived for a long time. Of these the *Bhuta Samkhya* system seems to be the older one.

The *Bhuta Samkhya* System : *Samkhya* means number and *Bhuta* element, part, component, number etc. In this system, numbers are indicated by well-known objects or concepts having as many parts or components as the numbers they connote eg.:

0 is denoted by *sunya* (void), *kha* (sky), *antariksa* (atmosphere), *purna* (whole), *randhra* (hole) etc.

1 is denoted by *sasi* (moon), *bhumi* (earth), *go* (cow) etc.

2. is denoted by *netra* (eyes), *bahu* (hands), *karna* (ears), *paksa* (moon's waxing and waning periods) etc. - each of which has a pair of members.

3. is denoted by *kala* (time - past, present and future), *loka* (heaven, earth and hell) etc. - each of which has a trio of components.

4. is denoted by *veda* (*rk, yajur, sama* and *atharva*), *yuga* (*krta, treta, dvapara,* and *kali*), *dik* or *dis* (directions - east, north, west and south) etc. - each of which has a quartet of constituents.

5. is denoted by *bhuta* (elements), *pandavas,* etc. - each of which has five members

* * *

12 is denoted by *mas* (months), *rasi* (signs of the zodiac) etc. - each having twelve members.

* * *

32 is denoted by *danta* (teenth)

and so on. Any synonym of a word denotes the same number.

The principle of place value was used but the mode of writing was from right to left; thus both *netra-kala-yuga* and *bahu-loka-veda* stand for 432 (not 234) ; *masa-kala-yuga-danta* denotes 324312 (not 123432). *Sunya-kha-purna-randhra-netra-kala-yuga* is an expression for the number 4,320,000.

Neither the name of the individual who orginated this technique nor the school which used it for the first time is known. According to Datta and Singh "a system of expressing numbers by means of words arranged as in the place-value notation was developed and perfected in India in the early centuries of the Christian era." [11].

Mathematicians and astronomers of India too composed their works in verse. The multiplicity of words, especially the synonyms, corresponding to each number gave them latitude to choose anyone, suitable for versification. the same number could be expressed by means of word-combinations in a variety of ways. To decode the expressions to their numerical forms one would have to be familiar with concepts such as four *yugas*, five *pandavas*, seven *sages* etc.

occurring in the holy books of the Hindus.

The same word sometimes stood for two and occasionally for more than two different numbers eg. *go* has been used for 1 as well as 9; *paksa* for 2 as well as 15; *loka* for 3 as well as *14; dik* for 8 as well as 10 (*dis* or *dik* also stands for 4). Further Mahavira used *ratna* to denote 3 while others used the same term to denote 5; *tatva* to Mahavira was 7 while to others it was 5 or 25 and so on

These ambiguities must have created some confusion and also gone against the universal acceptance and prolonged survival of the *Bhuta Samkhya* system. Occasionally the *Bhuta Samkhya* system has made its appearance in Keralese works belonging to later periods too. This system can now be viewed as a historic relic, to be met with in many works of the ancient period and rarely in works of later period.

The *Katapayadi* system: The starting point of this system of numeration is the set of letters of the Sanskrit alphabet. The sanskrit alphabet is given below:-

ROMAN TRANSLITERATION OF DEVANAGARI

VOWELS

Short :	अ	इ	उ	ऋ	ऌ	(and क)	
	a	i	u	ṛ	l		
Long :	आ	ई	ऊ	ए	ओ	ऐ	औ
	ā	ī	ū	e	ō	ai	au

Anusvara : ं = ṁ

Visarga : : = h

Non-aspirant : ऽ = '

CONSONANTS

Classified :	क्	ख्	ग्	घ्	ङ्			
	k	kh	g	gh	ṅ			
	च्	छ्	ज्	झ्	ञ्			
	c	ch	j	jh	ñ			
	ट्	ठ्	ड्	ढ्	ण्			
	t	th	ḍ	ḍh	ṇ			
	त्	थ्	द्	ध्	न्			
	t	th	d	dh	n			
	प्	फ्	ब्	भ्	म्			
	p	ph	b	bh	m			
Un-classed :	य्	र्	ल्	व्	श्	ष्	स्	ह्
	y	r	l	v	ś	ṣ	s	h
Compound :	क्ष्	त्र्	ज्ञ्					
	kṣ	tr	jñ					

(Sanskrit is the language of the scholarly section throughout India and Malayalam is the regional language of Kerala. All letters of the Sanskrit alphabet occur as such in the Malayalam alphabet; the subdivisions into vowels and consonants etc. and pronounciations are identical. The only difference is in the scripts; Sanskrit is written *in the Devanagari script while Malayalam has its own script. So this* system applies to the Malayalam alphabet also).

In this system the digits 1 to 9 and 0 are denoted by the consonants as indicated below:

1	2	3	4	5	6	7	8	9	0
k	kh	g	gh	ṅ	c	ch	j	jh	ñ
ṭ	ṭh	ḍ	ḍh	ṇ	t	th	d	dh	n
p	ph	b	bh	m					
y	r	l	v	ś	ṣ	s	h	ḷ	

(The letter ḷ peculiar to the Dravidian pronounciation denotes the digit 9). Besides n añd n, pure vowels not preceded by a consonant also denote 0 (zero). In the case of conjunct consonants only the last consonant has value. Vowels following consonants have no special value; so each of the letters k, ka, ki and ku denotes the same number, viz. 1).

The *Katapayadi* derives its name from k, t, p and y, the letters in the first in the first column of the table, given above.

In the *Katapayadi* system, which follows the decimal notation, a right to left arrangement is accepted i.e. the letter which stands for the digit in the unit's place of a number is written first, the letter which denotes the digit in the ten's place is written second and so on. Thus *Madhava* (*ma* = 5, *dha* = 9, *va* = 4) indicates the number 495 (not 594) ; *nanajnana pragalbha* connotes the number 4, 320, 000 since *na* = 0, *ra* = 2, *ga* = 3 and *bha* = 4.

A special advantage of this system is that numbers running to several digits can be rendered into meaningful expressions or versebits, which are easy to remember. The mnemonic of any number is called its *paralperu*. Moreover the variety in the choice of letters (consonants and vowels following them) for each digit makes it

possible to frame several different expressions for the same number. In brevity and conciseness the *Katapayadi* excels the *Bhuta Samkhya* system; compare the expressions, given above, for the number 4,320,000 in the two systems.

The general belief is that the astronomer Vararuci was the proponent of the *Katapayadi* system. From circumstantial evidence, historians of Malayalam literature surmise that Vararuci lived in the fourth century A.D. Some variants of the *Katapayadi* system can be found in the writings of Aryabhata II (c. 950 A.D.) as also in some Pali manuscripts in Burma. Priority in time goes to the *Katapayadi* system attributed to Vararuci. This scheme, known as the Kerala system of the *Katapayadi*, was used in Kerala from the time of Vararuci and in course of time it became popular throughout south India. the Katapayadi system is generally believed to be one of the major contributions of Kerala to Indian mathematics. [11; 31; 47 & 74].

Vararuci has to his credit another achievement, namely the composition of the *candra-vakyas.* the 248 candra-vakyas give the positions of the rising moon on as many consecutive days. They are couched in the *Katapayadi.* These candra-vakyas are used by Keralese scholars from the fourth century. [39, Appendix].

In the section Dasa-gitika of the *Aryabhatiya*, Aryabhata has given an alphabetic scheme for representation of numbers. This scheme is different from both the *Bhuta Samkhya* and the *Katapayadi* systems. It is more 'compact and concise'; but this advantage is more than offset by two disadvantages inherent in it. The first is that most of the letters formed here are very difficult, sometime impossible to pronounce. Second is that this system allows only limited choice of expressions for a number. This system seems to have been used only by Aryabhata. It is based on the letters of the Sanskrit alphabet. Explanation is complicated. We suggest readers interested in the system to refer: [11; 16; 31; 61; 96 or 98].

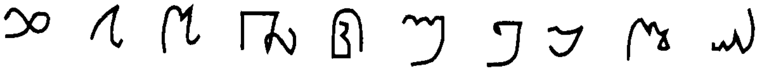
* * *

Kerala had a unique set of symbols to represent the numbers from 1 to 10:

1 2 3 4 5 6 7 8 9 10



A variant of these symbols



was used by some in Malabar i.e., northern Kerala. [100, Vol. II]

These symbols were in use in the schools till a few decades ago; even now some books are printed with their page numbers in these symbols. The Hindu numerals, 1, 2, 3, 4, 5, 6, 7, 8 and 9 have phased out these symbols from schools.

ĀRYABHATA

The most famous Indian mathematician of yore is undoubtedly the great Aryabhata (pronounced AH-RYA-BHATA). Many things about him - his place of birth, his date of birth and even his real name - are still subjects of controversy. The situation becomes all the more confusing because of the indications that more than one mathematician bearing the name Aryabhata flourished in India at various periods. That there were at least two Āryabhatas is beyond doubt: (i) Āryabhata of *Kusumapura* who wrote the *Āryabhatiyam* in 499 A.D. and (ii) Aryabhata who wrote the astronomical treatise called the *Ārya Siddhānta* also known as the *Mahā Ārya Siddhānta* (c. 950 A.D.) Brahmagupta (7th century), in his *Khandakadyaka* as also Al Biruni (973-1048) in his *India* indicate that another Āryabhata existed prior to Āryabhata of *Kusumapura*. Here we confine ourselves to Aryabhata of *Kusumapura*. That he was born in 476 A.D. is now an accepted fact but the exact date of birth remains unknown.

Kusumapura was not Āryabhata's place of birth : this has been borne out by the statement *Aśmakajanapadajāta Āryabhatācārya* (Āryabhata, the preceptor was born in *Aśmaka* country) contained in the commentary on Aryabhatiya, written by Nilakantha (1444 - 1545). It is therefore clear that Aryabhata was born in *Aśmaka*.

Here we present some comments regarding Āryabhata's places of birth, residence etc.

Scholars had thought for a long time that Āryabhata lived and wrote his *Aryabhatiya* at the city of *Kusumapura*. This city was identified with *Pataliputra* (modern Patna) in *Magadha* (modern Bihar). They argue thus: *Magadha*, in ancient times, was a great center of learning. The famous university of *Nalanda* was situated in this country in the modern district of Patna. There was a special provision for the study of astronomy in this university. An astronomical observatory is also said to have existed at *Khagaula* near Patna. The

name *Khagaula,* which is a corruption of the astronomical term *Khagola* meaning 'celestial sphere'. itself suggests that there must have been an astronomical observatory at this place. It is quite likely that Aryabhata was a *kulapa* or *kulapati* (head) of the University of *Nalanda* which flourished during the fifth to seventh centuries A.D.

Such a view now appears untenable in the light of recent studies on the works of some commentators of *Āryabhatiya*. In these works, Āryabhata is frequently referred to as an *Asmaka*, that is one belonging to the *Asmaka* country. In some quarters *Asmaka* is identified with Kerala state. Practically all the astronomers of the Āryabhatan school, whose places of origin can be definitely determined, belong to this part of India.

From atleast the seventh century A.D, if not earlier, Kerala had been bastion of Aryabhatan school of mathematics and astronomy. The Āryabhatan system had been extremly popular in Kerala. All the astronomical works produced in Kerala, whether commentatorial or original, follow the Āryabhatan system. Further the efforts of mathematicians and astronomers belonging to Kerala have generally been directed towards the revision of, additions to and corrections of the Āryabhatan system with a view to derive more accurate results. Most of the extant commentaries on the *Aryabhatiya* have been written by the astronomers of Kerala. Manuscripts of the work of this school are found mostly only in this part of the country.

H. Kern has stated that three manuscripts of the *Āryabhatiya* he could collect were written in the Malayalam script; while two of the manuscipts are dated 1820 and 1863, the date of the third manuscript is not recorded. Further the manuscript dated 1863 contains the statement copied from an *ola* manuscript in the Chirakkal Raja Library 'written by Unni Panikkar of Calicut 1863' (*Ola* is the Malayalam word for palm leaf; leaves of a particular variety of palmyra tree - not coconut palm - were dried, smoothened and cut into strips of uniform length; documents to be preserved were written in such strips in olden days).

We do not find an Aryabhatan school or its influence in Bengal or Patna. This leads us to surmise that *Kusumapura* might be a place in south India.

Another clue lies in the *Siddhānta* followed by Āryabhata. Five different versions of the *Siddhāntas* or systems (of astronomy) are known. They are the *Pauliśa Siddhānta*, the *Sūrya Siddhānta*, the *Vasistha Siddhānta*, the *Paitāmaha Siddhānta* and the *Romaka Siddhanta*. The *Paitamāha Siddhānta* is also called *Brahma Siddhānta* or *Swayambhuva Siddhānta*. The *Siddhāntas* were composed during the late four century or early fifty century. Of these, the *Sūrya Siddhānta* written about 400 A.D. is the only one that seems to be completely extant. Later writers report that the *Siddhāntas* were in substantial agreement on substance, only the phraseology varied. Hence we can assume that the others, like the *Sūrya Siddhānta*, were compendia of astronomy, comprising cryptic rules in Sanskrit verse with little explanation and without proof. People in almost all parts of India, other than Kerala, had accepted the *Sūrya Siddhānta* while from very early times the people of Kerala were following the *Paitamāha Siddhānta*. The introductory verses in *Gitikapāda* and *Ganita-pāda* i.e. parts I and II of the *Āryabhatiya* indicate that it is based on the *Brahma Siddhānta*, which had been highly respected by the people of *Kusumapura* (*Kusumapure'bhyarcitam jnānam* - verse I of *Ganita - padam*). This statement does not imply that Āryabhata was a native of *Kusumapura*. [6; 7; 30; 38; 58; 59 & 74]

Āryabhata has recorded the year of composition of the *Aryabhatiya* in *Kali* era. It was only in Kerala and to some extent in the bordering areas of the Tamil country that *Kali* era was in common use; elsewhere in India *Vikram* era and *Saka* era (founded in 58 B.C. and 78 A.D. respectively) were in vogue. *Kollam* era of Kerala, which is also in use now, was founded only in 825 A.D. Prior to that scholars and writers of Kerala were using *Kali* era. [40].

Some scholars hold the view that Ārybhata was not the real name of the author or the Āryabhatiya and that it was a title conferred

on the author of the *Āryabhatiya* for composing the major portion of his work in the *ārya metre.* (Barring the verses forming the invocation and introduction as also the conclusion, the first part of *Āryabyhatiya* contains ten verses written in the *gitika metre*, followed by a second part consisting of three sections : *Ganita-pāda, Kālakriya - pāda* and *Gola - pāda* of 108 verses in all, in the *ārya metre.* [58].

To some, the name Āryabhata is not acceptable; they insist on having it as Āryabhatta. According to them the name of the author of the *Āryabhatiya* was Ārya; he was a *bhattatiri* (a scholarly namputiri of Kerala) or a *bhattācharya* (a scholar of Bengal) - both of which can be abbreviated as bhatta.

It is quite possible that *Aśmaka janapada* and *Kusumapura* were sanskritized forms of the names of places of birth and residence respectively of Āryabhata. A suggestion has been put forward that Aśmakajanapada might be the sanskritised version of *Kodungallur* (Kodum-kall-ur), previously known as *Thiruvancikulam* or *Mahodayapuram*, which was the capital of the *Cera* kings. [10 & 45}.

Further investigations are necessary and some documents giving decisive evidence have to be ferreted out if the claim is to hold good; till then no final verdict can be given regarding the locations of *Aśmakajanapada* and *Kusumapura.*

Whether Āryabhata was a native, by birth or by domicile, of Kerala may remain a subject of controversy but the hold that the *Āryabhatiya* had on the astronomers and mathematicians of Kerala was without an equal. The *Āryabhatiya* seems to be one fountain head from which mathematicians and astronomers of Kerala not only drank deep but also drew their inspiration for many centuries to come.

The *Āryabhatiya* is a very well known treatise. A discussion of its mathematical contents may be unnecessary. We shall recall those results needed in our discussions as and when necessary.

BHASKARA I

Among the prominent commentators of the *Āryabhatiya*, Bhāskara (pronounced BHAH-S-KARA), is one of the ēarliest. (We shall designate this Bhāskara as Bhaskara I so as to distinguish him from the more famous twelfth century mathematician Bhāskara II, the author of the *Siddhānta Siromani.*) Bhāskara I was not a direct disciple of Āryabhata. He belonged to the school of Āryabhata and was one of its ablest exponents. That Bhāskara I identified himself with this school is quite apparent. He refers to Aryabhata as 'our preceptor' in several places in his commentary on the *Āryabhatiya*; further in one place this ardent follower of Āryabhata says that 'for us the four quarters of the yugas are equal' a scheme put forward by Āryabhata which was different from those held by others. The terms *Aśmaka*, *Aśmakiya* and *Aśmaka-tantra* or *Aśmaka-sutra* used by Bhaskara I to denote Āryabhata, followers of Āryabhata and the *Āryabhatiya* indicate a strong predilection that Bhāskara I had with Āryabhata. [38 & 94].

We do not have sufficient evidence to say anything definite about the place Bhāskara I hailed from or the place where he settled down and wrote his works. Dr. K.S. Shukla opines that there are, however, reasons to believe that Bhāskara I belonged to *Asmaka* country and that he lived and taught at *Valabhi* (modern Saurastra or Kathiawar) where he wrote his commentary on the Āryabhatiya. [97]. Scholars like A.N. Singh of Lucknow hold that there is sufficient justification to state that Bhāskara I was a native of Kerala. [44, Vol. I].

Another opinion runs as follows: "Stray references to his (Bhaskara I's) works appear to indicate his association with Saurastra (in western India) and *Aśmaka* (south India, probably Kerala). It is possible that he was a native of either of these regions and migrated to the other."[6].

Until now three works of Bhāskara I have come to light; they are (i) *the Mahā Bhāskariya* (ii) *the Laghu Bhāskariya* and (iii) *the Āryabhatiya bhāsya* (commentary on the *Āryabhatiya*). Though popularly known as the *Mahā Bhāskariya* and the *Brhad Bhāskariya* the name intended by the author for this work was the (*Brhad*) *Karma Nibandha* (a treatise on astronomical calculations). It contains an exposition of the material formulated in Āryabhatiya. Keralese astronomers look upon Āryabhata as a *sūtra-kāra* (i.e., a formulator) and Bhāskara I a *vrtti-kāra* (i.e., an expositor) The treatment of each topic in the *Mahā Bhāskariya* is fairly exhaustive; the author aims at clarity rather than conciseness and brevity; he also introduces alternate methods to make the material easy and clear to the reader. Bhāskara I has given a rational appromixation to sine function. In the language of modern trigonometry it can be displayed as:

$$\sin x = \frac{y(180 - y)}{(1/4)\,[40{,}500 - y\,(180 - y)]}$$

where y = x, 180 - x, x - 180 or 360 - x, according as x lies in the first, second, third or fourth quadrant respectively. [38] (For derivation of this result, see part II - Mathematical Section of this book.)

As a simple algebraic approximation to a transcendental function, this formula is good as the following table shows:-

Angle in degrees	sine according to this formula	sine from standard tables
0	0	0
5	.088	.087
10	.175	.174
20	.343	.342
30	.500	.500
40	.642	.643
45	.706	.707
50	.765	.766
60	.865	.866
70	.939	.940
80	.985	.985
90	1.000	1.000

The *Laghu Bhāskariya* is an abridged and simplified version of the *Mahā Bhāskariya.* Aware of the fact that the treatment and style adopted in the *Mahā Bhāskariya* are difficult for beginners, Bhāskara I, in his *Laghu Bhāskariya* has consciously simplified the language, dropped certain topics which are difficult and unnecessary for the novice etc.

It would be appropriate to say that the *Laghu Bhāskariya* is a well - planned summary of the important rules and processes contained in the *Mahā Bhāskariya,* set forth in a systematic and lucid manner.

The *Āryabhatiya - bhāsya* : This commentary of Bhāskara I on the *Āryabhatiya* was recognised as a work of great scholarship and its author was called *Sarvajña bhāsyakāra* (all knowing commentator).

It is interesting to note what Bhaskara I has to say of himself; he ends his commentary thus: "These aphorisms written by Aryabhata deal with matters beyond the reach of the senses. It is impossible for people like us to explain even one-hundredth part of their meaning, not to say of the whole." [97]

Bhaskara I's commentary contains a detailed and exhaustive discussion of the contents of the text. Rules of mathematics are illustrated by means of solved problems. Besides giving a detailed exposition of the textual material, the commentator has introduced and discussed topics which are either omitted or not explicitly stated in the *Aryabhatiya*. In order to make the subject matter clear and interesting, doubts are raised, questions and cross questions are put forward and appropriate answers given. While discussing the rule; "In the case of all plane figures, one should determine the adjacent sides (of the rectangle into which the figure can be transformed) and find the area by taking their product." (verse 9, *Ganita pada* of the *Aryabhatiya*), Bhaskara I argues thus:

DOUBT : If the areas of all (plane) figures can be determined by this rule then previously stated rules would become useless.

ANSWER : They are not useless. Both calculation and verification are taught by this rule. The areas calculated by previously stated rules have to be verified. The mathematicians Maskari, Purana, Putana etc. prescribe verification of all (plane) figures (by deforming them) into rectangular figures. So has it been said "having determined the area in accordance with the prescribed rule, verification should always be done by (deforming the figure into) a rectangle, because rectangle is the only figure, whose area is obvious." [97].

The works of Bhaskara I were highly appreciated in South India, particularly in Kerala and the neighboring states. Several commentaries came to be written by Keralese scholiasts such as Govindaswami (c. 800 - 850 A.D.), Sankaranarayanan (c. 825 - 900 A.D.), Udayadivakaran (eleventh century A.D.) and Paramesvaran

Namputiri (1360 - 1455 A.D.). Keralese scholars recognised the *Maha Bhāskariya* and the *Laghu Bhāskariya* as text books par excellence; this 'recognition' continued for centuries. Clarity of expression, lucidity of exposition and judicious arrangement made the *Laghu Bhaskariya* more popular especially with those beginning to study the subject. In addition to this, commentaries on the *Laghu Bhāskariya* are available not only in Sanskrit but also in provincial languages; this enhances and attests to its popularity.

The greatest testimony to the excellence of the writings of Bhaskara I is that they survived for more than thirteen hundred years. Although scores of text books have been written, none succeeded in replacing the works of Bhaskara I.

Study of Bhāskara I's commentary on the *Āryabhatiya* did not remain confined to South India. there are reasons to believe that it was popular in North India at an early date. Prithukaswami (c. 860 A.D.), who lived in Kanauj, in Uttar Pradesh, in his commentary on the Brahma Sphuta siddhānta of Brahmagupta refers to Bhāskara I and his interpretations of certain rules of Āryabhata as many as four times.

T.S. Kuppanna Sāstri fixes the limits to Bhāskara I's date between 550 A.D. and 628 A.D. Dr. K.S. Shukla, on the strength of internal evidence, points out that the (*Āryabhatiya - bhāsya*) was written in 629 A.D. It seems reasonable to fix Bhāskara I's date between 550 and 630 A.D.

Other personal details known about Bhāskara I are that he was a Hindu probably a brahmin and a devotee of Lord Shiva to whom he pays obeisance in his Mahā Bhāskariya and Āryabhatiya-bhāsya; he imbibed his knowledge of astronomy from his father. [38 & 94].

As in the case of Aryabhata we cannot decide whether Bhāskara I was a native of Kerala by birth or domicile; but the influence that his works had on Keralese astronomers and mathematicians is next only to that of the *Āryabhatiya*.

MID - SEVENTH to MID - FOURTEENTH CENTURIES

Keralese works belonging to the next seven centuries (i.e. the period from 630 A.D. to 1340 A.D. to be specific) which have been salvaged so far contain very little of innovative mathematics. Scholars such as Haridatta (650 - 700), Govindaswami (800 - 850), Sankaranarayanan (825 - 900), Krisnacarya (1200), Suryadeva yajvav (1191- 1250), Govinda Bhattatiri of Talakkulam (1237 - 1295) to mention only a few, enriched astronomical and astrological sections of *Jyotis'-sastra* by their contributions. Their efforts were directed to the production of commentaries, improvements etc. on the works of earlier scholars.

With the efflux of time, inaccuracies were detected in some results in the Āryabhatiya positions of planets reckoned by Āryabhatan methods did not agree with their observed positions. A system of calculation called the *Parahita* was promulgated in 683 A.D. It applied to 'planets' rather than the sun, which was considered a planet as was the moon. *Grahacaranibandhana* of Haridatta is the basic manual of this system. Haridatta based his system on the *Aryabhatiya* but improved it in several ways. Easier methods replaced tedious calculations; simpler *katapayādi,* the difficult Āryabhatan scheme. The revised system brought the high-browed astronomical discussion and calculation to the level of the ordinary. *Hita* means liking and *para,* one outside the fold (here of astronomers) i.e. layman. [68 & 75].

Para is also abbreviation for *paraloka* i.e., world of the dead. The Brahmin community is a ritual - laden group. It has some kind of a ritual, religious or social, for almost anything and everything. The saga of rituals pertaining to an individual starts months before birth runs through life and continues for years after death. No individual can perform all his rituals, obviously. His father or eldest son, usually does the pre-birth or post-death ones respectively for him. Women

are prohibited from performing them. Rituals, like birthdays, death anniversaries are conducted on days linked with asterisms, the crescent of the moon (in the waxing and waning periods) respectively; others like *upanayanam* (the initiation of a boy to brahmin way of life), marriage etc., within specified time-limits (or *muhurtams*) on pre-determined dates. A knowledge of the exact positions, correct movements etc. of celestial bodies is necessary to fix the dates and time-limits for rituals. According to religious preceptors, a ritual performed 'outside the correct time' would not have the desired effect. The brahmin community welcomed revisions, refinements etc. on the methods of astronomical calculations, allowing for greater precision and adherence to their religious tenets. Parahita system gained immediate acceptance and popularity in Kerala. Gradually it spread to other regions (covering the present Tamil nadu and Andhra Pradesh).

Sankaranarayanan (825-900 A.D.) was a contemporary and protege of the *Cera* king Ravi Varma Kulasekharan. Mahodayapuram, the capital of *Cera* kingdom, was in a prosperous condition at that time. Ravi Varma took keen interest in astronomical studies and held discussions with astronomers. Sankaranarayanan was the astronomer - royal, in charge of the observatory located at Mahodayapuram. [9 ; 47 & 74].

Predictional part of *Jyotis' - sastra* or astrology soared to great heights. Govinda Bhattatiri was a very bright star in the galaxy of Keralese astrologers. A line of astrological tradition he started continues to this day. [64].

KERALESE CALENDAR

Here we take a detour. The year 825 A.D. was a historic one for Keralese astronomers. By mid-August of the year a new era was launched. It is called *Kolla - varsham* in Malayalam and Malabar Era (abbreviated M.E. in English). Ravi Varma Kulasekharan was the *Cera* king at that time. He was a mathematician and the author of a mathematical treatise. According to some researchers, an additional era called *Parasurama* Era commenced in 1176 B.C. Stray references to this era occur in some works; details of the mode of reckoning and applications are not available.

Ancient astronomers of many civilizations observed that the moon and planets were never at very great angular distances from the ecliptic (apparent annual path of the sun); They imagined a belt in the sky extending to eight degrees on either side of the ecliptic and called it the zodiac. Twelve groups of stars or constellations were spotted in zodiac; further they noticed that the sun took a month to 'traverse' each constellation. The constellations were named according to their shapes as Aries (ram), Taurus (bull), Gemini (twins), Cancer (crab), Leo (Lion), Virgo (virgin), Libra (scale), Scorpio (scorpion), Saggitarus (archer), Capricornus (goat), Aquarius (water-bearer) and Pisces (fish). These are known as 'signs of the zodiac'. We come across this list, with some variations, in several civilizations. Some countries adopted these 'signs' as names of the twelve months of the year; Kerala was one, as the following table shows.

An analysis of Keralese calendar for 105 years (1061 M.E. to 1165 M.E.) gives average durations of different months as follows:-

Name of month: Gregorian calendar	Name of month: Keralese calendar	meaning of the word denoting name of month	Duration in days (correct to 2 dec. places)
August – September	Chingam	Lion	31.03
September – October	Kanni	Virgin	30.46
October – November	Tulam	Scale	29.93
November – December	Vrischikam	Scorpion	29.54
December – January	Dhanu	Bow	29.39
January – February	Makaram	Sea-monster	29.49
February – March	Kumbham	Water-pot	29.83
March – April	Meenam	Fish	30.35
April – May	Medam	Ram	30.90
May – June	Edavam	Bull	31.33
June – July	Mithunam	Twins	31.58
July – August	Karkatakam	Crab	31.42
			365.25

Note the variations of words : 'bow' for 'archer', 'sea-monster' for 'goat' and 'water-pot' for 'water-bearer'.

Keralese term for 'sign' is *rāsi*. Each *rāsi* was, to them, an arc of a circle facing an angle of 30 degrees at the center. They calculate the time of 'entry' of the Sun in each *rāsi*. The interval between the times of entry into two consecutive *rāsis* is a (theoretical) month. The 'entry' into a *rāsi* can occur at any time of the day but a month cannot. For practical purposes they have to follow some conventions. One is that a month begins on a day if the 'entry' takes place before noon,

otherwise the next day. Table given below shows average durations of 'theoretical months' calculated by Keralese astronomers. Calculation is too involved and elaborate to be included here. This table was supplied by an eminent calendar maker Prof. P.U. Krishna Variyar of Kottayam, Kerala, to whom thanks are due. The durations are subject to an error of up to 9 minutes because of perturbations:-

Name of month	Duration in			
	Days	hours	monutes	Days (in decimals)
Chingam	30	23	51	30.99375
Kanni	30	11	48	30.49166
Tulam	29	23	38	29.98472
Vrischikam	29	14	31	29.60486
Dhanu	29	10	39	29.44375
Makaram	29	12	59	29.54097
Kumbham	29	20	58	29.87361
Meenam	30	08	38	30.35972
Medam	30	21	02	30.87638
Edavam	31	06	44	31.28055
Mithunam	31	10	57	31.45625
Karkatakam	31	08	24	31.35000
	365	06	09	365.25622

Both tables show that five consecutive months *Tulam, Vrischikam, Dhanu Makaram* and *Kumbham* each has less than 30 days on the average and that seven months *Meenam, Medam, Mithunam, Karkatakam, Chingam* and *Kanni* each has more than 30 days on the average (a majority close to or more than 31 days); moreover the tables show that the durations of months increase from *Dhanu* to *Mithunam* and then decrease from *Mithunam* to *Dhanu*, completing a cycle every twelve months.

Now we can explain this phenomenon invoking the first two laws of planetary motion enunciated by Kepler (1571 - 1630 A.D.) namely : (i) each planet moves in an elliptical orbit with the sun at one of the foci and (ii) the straight lines drawn from the sun to a planet (i.e., the planet's radius vector) sweeps equal areas in equal times. How did Keralese astronomers design such a calendar, more than seven centuries before Kepler?

We now return to our main journey along the highway of mathematics.

THE GOLDEN AGE

The period 1350 A.D. - 1600 A.D. was the glorious period Kerala Mathematics attained great heights. Study of the circle and its chords broke the 'finite' barrier and soared into the 'infinite'.. It was a rewarding venture. There lay a superb mine of precious things, the like of which the world had not seen before. The pioneers reaped a rich harvest. They couched the valuable items, the 'end products' of their explorations, in nice poetic setting and arranged them in palm leaf manuscripts. Most of the innovators never cared to give even an inkling of their labors. These manuscripts found their way into ill-ventilated dungeons, known by the name *grandha-puras* (store-house of books and manuscripts) of palaces, illams (or traditional namputiri houses) etc. These private collections were accessible only to the select. Those who could assess their worth went on to clarify or improve the material therin. Nobody came forward to publicize the knowledge contained in the hidden treasure. In short the world was in the dark about the achievements of Keralese mathematicians during the period mentioned above. We shall call this period the *golden age* of Keralese mathematics.

No wonder historians of mathematics declared that for several hundred years after the twelfth century A.D., India contributed almost nothing to the mathematical treasure.

C.M. Whish (18th - 19th centuries) was the first foreigner who tried to draw the attention of the outside world to Keralese mathematics of the golden age. Here is an account of what Whish did: Charles M.Whish (of the Madras establishment of East India Company civil service) visited Kerala during the early part of the nineteenth century and collected some antiquities. The information he could gather about them was summarized and put on labels, which he attached to these curious looking items. These labelled curios were arranged in a showroom and the showroom was officially opened on December 15, 1832. The public had to wait for over two years to get a glimpse

of Whish's collection.

Sight-seers who passed by Whish's showroom were unaware of the worth of these exhibits and so never bothered to cast their eyes a second time in that direction. Thus the explorative and organisational labours of Whish remained unrecognised and unsung for over a century.

By the 1940s, occasionally an enthusiast or two wandered into Whish's "old curiosity shop". The specimens kept there amzed them and roused their interest and enthusiasm. Gradually, more people were drawn to these antiquities and this led to intensive and extensive investigations; by the eighth decade of this century, Whish's small showroom had become a nucleus around which activities were toning up and spreading out.

This, in short, is the story of the development of Whish's work on the mathematics of Kerala. What was referred to as Whish's showroom is an article titled "*On the Hindu Quadrature of the Circle and the infinite series of the proportion of the circumference to the diameter exhibited in the four sastras, Tantra Sangraham, Yucti Bhasa, Carana-Paddhati and Sadratnamala*". This paper was read at the Royal Asiatic Society of Great Britain and Ireland on December 15, 1832 and published in its *Transactions*, vol. III, 1835.

Very rarely does a book or an article on science remain valid in toto after the lapse of a century. Most of the publications need amendments to or even overhauling of sections. These are necessitated by new discoveries resulting from investigations and explorations conducted in the meanwhile. Whish's article is no exception. Without any doubt, Whish deserves our gratitude and praise for his pioneering work of an explorative nature, which enable him to reveal a 'treasure trove" lying hidden for generations.

It took more than twelve decades for Keralese mathematics to make an inroad into the domain of books. Dr. C. . Srinivasaiengar's *History of Ancient Indian mathematics* was one of

the earliest. About Indian mathematics after the twelfth century, he says: "The development of mathematics came practically to a full stop, and scholoars contended themselves with chewing the cud, studying the works of great mathematicians, and producing here and there a small bright gem. This remark holds for the entire country, except the state of Kerala, in the south-west corner of India..... Some mathematical works of this place dating from the fifteenth to the seventeenth century have recently come to light and contain mathematics of a standard which is startling, and which sets a big puzzle to the historian..... They are essentially astronomical treatises but they give in accurate form what we now call Gregory's series, and Euler's series for π, and a number of remarkable rational approximations and rapidly convergent series for π." [102].

Researchers and editors such as C.T. Rajagopal, K.S. Shukla, T.S. Kuppanna Sastri, T.A. Saraswathi Amma, R.C. Gupta, K.V. Sarma, Rama Varma (Maru) Tampuran, Akhileswara Aiyar, P.K. Koru etc. have, directly or otherwise gleaned what they could from the *grandhapuras* of Kerala. Our attempt here is to consolidate what these and other enthusiasts gathered.

MADHAVAN

Kerala, in her heyday, gave birth to several scholars and men of genius. Madhavan (pronounced MAH-DHAVAN) of Sangamagramam was the morning star of the golden age. He deserves an important niche. Till recently the scholars outside Kerala had not heard of him. This is no surprise. Three decades back, very few people in Kerala, his native land, knew about him. They could at the most repeat that "Mādhavan is the author of the *Venvāroha* and also the discoverer of the *Jive-paraspara-nyaya*". [Jive-paraspara-nyaya in the parlance of Āryabhatan school meant 'rule to expand sin (A + B)']. A number of Indian mathematicians e.g., Aryabhata,

Bhaskaras, Brahmagupta, Mahavira to mention only a few, made significant contributions to arithmetic, algebra, mensuration etc., but it was Madhavan (1340 - 1425) who 'took the decisive step onward from the finite procedures of ancient mathematics to treat their limit passage to infinity, which is the kernel of modern classical analysis'. [52].

Not much is known of Madhavan's personal details. What little we know, is the result of weaving together the stray strands gathered from the works of later writers.

Sangamagramam derived its name from Sangamesvaran, the deity of the celebrated temple situated there. It has been identified with modern Irinjalakuda (Irinnalakuda) in central Kerala. This Madhavan is called 'Madhavan of Sangamagramam' (or Venvaroha Madhavan) to distinguish him from other Keralese Mādhavans like the astrologer Vidya-Madhavan etc.

The *Venvaroha* is an important work of Mādhavan. It is available in print now. It says that Madhavan belonged to the house called *bakuladhisthita vihara*, which is the sanskritized form of the Malayalam name *iranni ninna palli.* bakulam means iranni (the tree *mamusops elengi*) and *vihara* stands for *palli.* At present there is no house bearing this name at Irinnalakuda ; but there is a house named *irinnalapilli* (also called *irinnadapilli*) near Kallettumkara railway station, about eight kilometers from Irinjalakuda. The original name *iranni ninna palli* might have undergone some transformations and ended up as *irinnalapilli.* [44, vol. II; 77 & personal enquiries].

Later writers refer to Madhavan as *golavid* (adept in spherics). An old document reveals that Mādhavan belonged to a sub-caste of brahmins known as *emprantiris*, a priestly class. [74].

We have no information about his parents and/or preceptors.

There are two works of Madhavan the *Sphuta-candrāpti* and the *Aganita* which contain their dates of composition; the dates correspond to those in 1400 A.D. and 1418 A.D. respectively. No

other direct evidence about his time is available now. Nilakantha Somayaji (1444 - c. 1545 A.D.) in his commentary on the *Āryabhatiya*, explicitly mentions that Paramesvaran Namputiri (1360 - 1460), well-known as the propounder of the Drgganita, was a disciple of Mādhavan. Paramesvaran finalised the Drk system (the *drgganita*) in 1431 A.D. There is a report that when Parameśvaran brought the Drk system to the notice of Mādhavan, the latter suggested that the new scheme may be included in the section on mathematics (of Jyotis-sāstra) and not to use it for astrological purposes. If this report is true, then Mādhavan ought to be alive at the time Drk system was taking shape though not finalised. Mr. K.V. Sarma surmises that Mādhavan lived between 1340 and 1425 A.D. [44, vol. II; 74 & 85].

Besides the *Venvāroha*, the *Sphutacandrāpti* and the *Aganita* mentioned above Mādhavan is known to have written the *Lagnaprakarana*, the *Madhyamanayana prakara*, the *Mahājyānayanaprakara*, the *Aganitapan-changa*, and the *Aganitagrahacara*. Another work the *Golavāda* of Madhavan helped stabilize his appellation *Golavid*. These works deal with astronomical topics. A number of aphorisms of Mādhavan occur as quotations in later works. They are just statements of mathematical results. Mādhavan's work, if any, containing the aphorisms remains to be unearthed. We have as yet no first hand information about Madhavan's discoveries and/or demonstrations of those results.

We now proceed to the mathematical results 'discovered' by Mādhavan, as revealed by later works:-

Popular belief was that Mādhavan discovered *Jive paraspara nyāya* i.e., the result:

$\sin(A + B) = \sin A . \cos B + \cos A . \sin B$. [58 & 89]

This belief was a misplaced one. Madhavan was not the first mathematician to enunciate this result - not even the first Indian. The famous Bhaskarācārya (1114 - 1185 A.D.) had done it before. See verses 21-22 of the *Jyotpatti*, which is part three of Bhāskaracārya's

well known work the Siddhānta Siromani (1150 A.D.).

The world of mathematics had this result about a millenium before. The first book of *Almagest* compiled by Claudius Ptolemy (100 - 178 A.D.) contains the elegant theorem still known as Ptolemy's theorem namely: "In a cyclic quadrilateral, the product of the diagonal is equal to the sum of products of the two pairs of opposite sides." and derivation of some trigonometrical formulae including that of sin (A + B). It is believed that Ptolemy's *Almagest* owes much to the work of Hipparchus (180 - 120 B.C.) [3; 7; 91 & 105].

"It is believed, however, that the elegant theorem generally known as Ptolemy's theorem is due to Hipparchus and was copied from him by Ptolemy and that it contains implicitly the formulae for sin (A ± B)." [5].

Our knowledge of Hipparchus' achievements is second hand for nothing of his writings has come down to us. [15].

Mādhavan's *Jive paraspara nayāya* may, perhaps, be an independent re-discovery. Mādhavan was a lone adventurer wending his way in the mathematical domain without the help of a guide or preceptor worth the name. The problems he handled were universal. So it is inevitable that some of Mādhavan's work should prove on examination to have been anticipated.

Mādhavan has given the value of π correct to elevan decimal places. It is couched in the *Bhuta Samkhya* system. The verse runs thus:-

33 2 8 8 3

Vibudha-netra-gaja-ahi-hutasana

3 3 4 27 8 2

tri-guna-veda-bha-vārana-bāhavah

9 x 10^{11}

nava-nikharva-mite vrtivistare

paridhi-manam idam jagadur budhah

i.e., for a circle of diameter 9×10^{11} units, the circumference is 2,827,433,388,233 units. This gives the (approximate) value of π as 3.14,159,265,359 - a refinement. [57; 79 & 93]

Now we present the real discoveries of Madhavan : -

I. $$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots\ldots\ldots\ldots$$

II. $$\tan^{-1}x = x - \frac{1}{3}.x^3 + \frac{1}{5}.x^5 - \ldots\ldots\ldots\ldots$$

This result is known as Gregory's series. The Scotch mathematician James Gregory (1638 - 1675) published his work titled *Vera Circuli et Hyper-bolae Quadratura* in 1667. It contained very significant results including the expansion in series of certain trigonometrical functions, both direct and inverse. [7; 91 & 100, vol. I].

The following quotation from Howard Eves' book may be of interest: "Not noted by Gregory is the fact that for x = 1, the series becomes

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots\ldots\ldots\ldots$$

This very slowly converging series was known to Leibniz (1646 - 1716) in 1674" [15].

Keralese mathematicians first proved the series for x = 1 and derived Gregory's series from it.

III. $$\frac{\pi}{\sqrt{12}} = 1 - \frac{1}{3.3^1} + \frac{1}{5.3^2} - \frac{1}{7.3^3} + \ldots\ldots\ldots\ldots$$

IV. Here are two approximations or remainder terms:-

$$\frac{\pi}{4} \sim 1 - \frac{1}{3} + \frac{1}{5} \mp \ldots\ldots \pm \frac{1}{2n-1} \mp \frac{n}{(2n)^2+1}$$

$$\frac{\pi}{4} \sim 1 - \frac{1}{3} + \frac{1}{5} \mp \ldots\ldots \pm \frac{1}{2n-1} \mp \frac{n^2+1}{[n^2+1)4+1]n}$$

V.

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots\ldots\ldots\ldots$$

and

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots\ldots\ldots\ldots$$

(How Keralese mathemathematicians derived these results is discussed, in detail, in Part II - Mathematical Section of this book.)

The reputation of a scientist cannot be made by rediscoveries or refinements; it must rest primarily and rightly on original achievements. Re-discoveries can be made by the talented. Refinements can emerge from the shrewd and intelligent. Innovations can spring only from creative minds. Of Mādhavan's varied contributions to Jyotis'-sastra, the outstanding ones concern the infinite-series expansions and the finite-series approximations (given above). In this field he possessed extra-ordinary intutition and ability making him a creative mathematician of a very high order. Madhavan was perhaps the first mathematician to reach penetrating conclusions in the branch of mathematics known as, infinitesimal analysis. Undoubtedly he merits an abiding place among the great mathematicians of the world.

With the exit of Mādhavan, Keralese mathematics lost a first rate luminary. But it did not plunge into complete darkness. Another luminary had appeared on the horizon. The new one was not so bright but it could keep the darkness away. The new star was Paramesvaran namputiri.

PARAMESVARAN NAMPUTIRI

Paramesvaran (pronounced PARA-MESH-VARAN) Namputiri was a *Rg-vedic* brahmin of *bhrgu gotra* and as*valayana sutra*. (*gotra* indicates the progenitor or ancestor and sutra the preceptor). He belonged to the village of Alattur in Ponnani taluk (of the present Malappuram district), Kerala. His house was called Vatasseri (Vatasreni in Sanskrit) *illam*. The *illam* stood on the northern bank of the river Nila or Bharatapuzha close to where it merges with the Arabian sea.

[44, vol. II; 78 & 81]

Details about Paramesvaran's parents are not available. His grand-father, a disciple of the famous astrologer Govinda Bhattatiri of Talakulam, was an astronomer of repute.

Paramesvaran refers to one of his teachers, Rudra, in several of of his works. Nilakantha Somayaji who was a student of both Paramesvaran and his son Damodaran, mentions about two other teachers of Paramesvaran viz. Narayanan and Madhavan. Narayanan was the son of one Paramesvaran; nothing more is known of him. Madhavan was the great Madhavan of Sangamagramam. [36, 74 & 83].

Paramesvaran was a prolific writer. He has to his credit several original works including the *Drg-ganita*, the *Gola-dipika*, the *Grahana - mandana*, the *Grahanastaka*, the *Grahana-nyaya-dipika*, the *Candracchayaganita* and the *Vakyakarana*. He has written commentaries on many standard works:- the *Bhata-dipika* (on *Aryabhatiya* - though succinct, very useful), the *Karma-dipika* (on the *Maha Bhaskariya* of Bhaskara I), the *Siddhanta - dipika* (on Govindasvami's commentary on the *Maha Bhaskariya*), the *Paramesvari* (on the *Laghu Bhaskariya of Bhaskara I*), the

Paramesvara (on the *Laghumanasa* of Munjala), the *Vritti* (on his own *Gola-dipika*), the *Vivarana* (on the *Suryasiddhanta*), the *Vivarana* (on the *Lilavati* of Bhaskara II) etc. Paramesvaran is known to have written some treatises and commentaries on works on astrology too.

The *Drg-ganita* is Paramesvaran's magnum opus. Prior to the advent of the *drk-system* (expounded in the *Drg-ganita*) the system of astronomical computation that prevailed in Kerala was the *Parahita*. The necessity for revising the *Parahita* system became apparent. Due to the accumulation of minute variations over a long period, the results of computation were found to differ appreciably from those of actual observations. It was under these circumstances that Paramesvaran started on his investigations. He evolved the *Drk*-system as a result of his astronomical studies coupled with observations of celestial bodies for over fifty-five years. Observations stretching over long periods are necessary for ascertaining the minute differences in planetary motion and other celestial phenomena. Only then can one draw valid conclusions, enunciate new rules for calculation and offer corrections to the existing schemes [71 ; 73 & 81].

With the advent of the new system, both the systems (the *Drk* and the *Parahita*) came to be used simutaneously. The *Drk* being more accurate, was used for casting horoscopes, prediction of eclipses etc. while the *Parahita*, being the orthodox system, continued to be used for fixing certain rituals and social functions. Gradually many people switched over to the *Drk* for all events. [44, vol. II & 81].

Both the *Parahita* and the *Drk* systems deal with methods of astronomical calculations. They offer nothing new to mathematics.

Paramesvaran had observed and recorded several solar and lunar eclipses from *Saka* 1315 (corresponding to 1393 A.D.) onward. He composed the *Drg-ganita* in 1431 A.D. So he ought to have commenced his investigations by 1376 A.D. If we assume that he was born in 1360 A.D., we may not be far off the mark. Paramesvaran has

recorded in his *Gola-dipika* that, it was composed in 1365 Saka Era (ie. 1443 A.D.) Nilakantha Somayaji was born in 1443 A.D.; he was a regular disciple of Damodran, son and student of Paramesvaran. In his commentary on the *Aryabhatiya*, Nilakantha says that Paramesvaran, his *paramacarya* (supreme teacher), explained to him certain points in astronomy. This implies that Paramesvaran was alive when Nilakantha was a student. So we can conclude that Paramesvaran flourished between 1360 A.D. and 1460 A.D.

[38; 47; 78 & 81].

We now proceed to enunciate a formula for the radius of the circle circumscribing a cyclic quadrilateral in terms of its sides. This occurs in Paramesvaran's commentary on *Lilavati* of Bhāskara II.

If a, b, c, d are the sides, s = (a + b + c + d) / 2 i.e. the semiperimeter of a cyclic quadrilateral and R, the radius of the circle circumscribing it, then

$$R = \frac{1}{4}\sqrt{\frac{(ab+cd)(ac+bd)[ad+bc)}{(s-a)(s-b)(s-c)(s-d)}}$$

Historians of mathematics give the credit for the discovery of this result to Professor Simon Antonine Jean Lhuilier (1750 - 1840) of Geneva. He published it in 1782 A.D. [5 & 100, vol. II]

Damodaran of *Vatasreni* (c. 1410 - c. 1510 A.D.) was an erudite scholar in *Jyotis'-sāstra*. He not only inherited interest and enthusiasm but also received good motivation in celestial studies from his father and teacher Paramesvaran. Nilakantha Somayaji, his disciple, quotes several excerpts from Damodaran's works. So far no work of Damodaran has come to light. Anyhow Damodaran can be considered a link connecting Paramesvaran to Nilakantha in the teacher-student chain carrying on the torch of celestial studies.

NILAKANTHA SOMAYAJI

The *Tantra Samgraha* is a well-known treatise. Nilakantha (pronounced NEELA-KANTHA) was a great scholar in *Jyotis'-sastra.* His fame spread over Kerala and the neighbouring Tamil-speaking area. But people took sometime to identify Nilakantha as the author of the *Tantra Samgraha.* Nilakantha is usually referred to with the appellation Somayaji, Somasutvan or Comatiri; the last word is the Malayalam derivative of the first two Sanskrit words. Somayaji means one who has performed Soma Yajna (or sacrifice). A colophon occurring at the end of Nilakantha's commentary on the *Aryabhatiya* contains the following information.: "Nilakantha hailed from Srikunda-gramam, belonged to the *Gargya gotra,* was a follower of the *Asvalayana sutra* of the *Rg-veda* and a *bhatta.* He was a performer of *SomaYajna.* His father's name was Jatavedan and he had an younger brother by name Sankara."

Srikunda-gramam is the sanskritized form of the Malayalam word Tr-k-kandiyur. This place is situated near Tirur railway station in Ponnani taluk (of the present Malappuram District), Kerala. During the goldent age of Keralese mathematics Tr-k-kandiyur was famous as a center of learning. Nilakantha's illam was called Kelallur - sometimes spelt as Kerallur. The original name of the house was Kerala-nall-ur (sanskritized as Kerala-sad-gramam). In course of time the name got abbreviated to Kelallur. Nilakantha is often referred to as Kelallur Comatiri. His full name is Gargya - Kerala-Nilakantha Somayaji. It may be noted that the word Kerala prefixed to Nilakantha's name refers to his house, not the state. The house was situated on the southern side of, not far away from the local temple. It has come to light that one Etamana illam now stands where Kelallur illam was. This is corroborated by the information that Nilakantha's family became extinct and the ownership of the family estates passed on to

the next of kin viz. members of the Etamana illam. [44, vol. III & 55].

Nilakantha's favorite deity was Lord Shiva installed in the famous temple at Trpparannode (sanskritized as Sri Parakroda), near his village.

Kausitaka-Adhya-Netra-Narayana, known locally as *Azhvanceri Tamprakkal* (pronounced AH-ZH-VAH-N-CHERI TAMPRAH-KAL), was the (hereditary) religious head of namputiris of Kerala. He was a patron and close friend of Nilakantha. The patron was not only interested in but also well informed in *Jyotis'-sāstra*. He had great esteem for Nilakantha's scholarship. The patron and protege used to hold discussions on some of the difficult topics in astronomy. The credit for enthusing Nilakantha in his astronomical investigations and promoting him to write monumental works like the *Tantra Samgraha* and a commentary on the *Āryabhatiya*, goes to Netra Narayana. [74].

Damodaran Namputiri, son of Paramesvaran Namputiri was the teacher mainly responsible for initiating Nilakantha to the study of *Jyotis'-sāstra*. Paramesvaran and Damodaran were residents of Alathur village. Nilakantha's village was close by. The traditional system of education that prevailed in India was known as *gurukula vidyābhyasam*. *Gurukula* means teacher's family and *vidyābhyasam*, education. Students reside with the teacher and his family for many years and receive instructions. They help the family in domestic chores, while not engaged in studies. That was part of their life. No difference is made between the king's sons and poor man's sons. The students come out of the training fully equipped to face life. Nilakantha stayed with Damodaran for his education. Paramesvaran Namputiri occasionally taught his sons's disciples. We have already mentioned that Nilakantha considered Paramesvaran as his supreme teacher. Nilakantha received instructions in *Jyotis'-sāstra* from one Ravi too. [59; 76 & 88].

Nilakantha travelled almost the entire state of Kerala. This was to collect manuscripts etc. from various sources. He was also in correspondence with scholars outside Kerala. The work titled *Sundara*

raja-prashottara is a collection of Nilakantha's clarifications on the doubt raised by Sundara-rajan, an astronomer of Tamilnadu. [47].

Besides the *Tantra Samgraha* and a commentary on the *Aryabhatiya*, Nilakantha has authored ten or more works. They include the *Gola-sara* (quintessence of Spherical astronomy - a primer), the *Siddhanta-darpana* (the mirror of the laws of astronomy), the *Candracchaya-ganita* (computation of time from gnomonic shadow cast by the moon and vice versa), the *Grahana-nirnaya* (computation of solar and lunar eclipses), the *Sundararaja-prasnottara* (mentioned above), the *Graha pariksākrama* (principles and methods of verification of astronomical computation with direct observations), a commentary on the *Candracchāya-ganita*, an auto-commentary on the *Siddhānta-darpana*, the *Grahanadi-grandha*, the *Jyotirmimamsa* (investigations on astronomical theories). It is probable that Nilakantha wrote more works than those mentioned above. Later writers quote Nilakantha, but these excerpts do not occur in the afore-said works. These works deal with astronomical topics. We shall return to a discussion of the contents of the *Tantra Samgraha* and the commentary on the *Aryabhatiya* a little later.

The extant works of Nilakantha bear out the intensive as well as extensive investigations conducted by him; they reflect also his erudition and deep insight in the science. Lucidity of exposition is the hall mark of his works. His scholarship was encyclopaediac. Sundararajan calls him *sad-darsani-parangata* i.e., one who has mastered the six systems of Philosphy. His knowledge of scriptures, sastras etc. was remarkable. *Jyotis'-sastra* was his forte.

Nilakantha has given his date of birth in the *Siddhanta-darpana* and its auto-commentary. In the *kali* era it is *'tyajamyajnatam tarkaih'*, which in the *Katapayadi* denotes the number 1,660,181. This works out to June 14, 1444 A.D. Nilakantha lived to a ripe old age, to be a centenarian. This is attested by a contemporary reference made of him in a Malayalam work on astrology, viz. *Prasna-sara* of Madhavan, a namputiri of *Incakkazhva illam* in Kerala. In this work written in

1542-1543 A.D., the author says that he could count upon reputed authorities like Kelallur to recommend his work to the public. So we can conclude that Nilakantha lived during the period 1444-1545 A.D.

We now return to the *Tantra Samgraha*. From internal evidence we can conclude that it was written in 1500 A.D. The *Tantra Samgraha* is a full-fledged text on astronomy. The work contains 431 verses divided into eight chapters. The author has included several mathematical results. One is a systematic codification of the rules for the solution of the astronomical triangle. The spherical triangleformed by the sun, north pole and zenith on the celestial sphere is called an astronomical triangle. The six elements namely three sides and three angles of a spherical triangle are not independent. Nilakantha states: "Given three, out of the five elements, two others can be calculated. This can be done in ten ways." He calls it the *dasa prasna* (or ten problems). The rules for the solution of the ten problems are given one-by-one in the *Tantra Samgraha*. The rules, on simplification, reduce to the basic formulae

(i) cosine rule :

$$\cos a = \cos b . \cos c + \sin b . \sin c . \cos A.$$

(ii) sine rule :

$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C}$$

and

(iii) contangent rule;

$$\cot a . \sin b = \cos b . \cos C + \cot A . \sin C$$

where a, b, c are the sides and A, B, C the angles of the spherical triangle.

The *Tantra Samgraha* is not the first Indian treatise to contain the rules for the solution of astronomical triangles. Some of these

ideas can be traced back to earlier works like the *Surya Siddhanta*, the *Siddhanta Siromani* etc. Nilakantha was, perhaps, the first Indian mathematician to collect and codify the rules. Even in the matter of codification Nilakantha was not the first in the world of mathematics. Johannes Muller (1436 - 1476 A.D.), better known as Regiomontanus, compiled a systematic account of the methods of solving triangles, both plane and spherical, by about 1464 A.D. This work titled *De Triangulis Omnimodis libri quinque* was published posthumously in 1533 A.D. [7 ; 15 & 91}

Next we turn to the *Aryabhatiya-bhasya* or commentary on the *Aryabhatiya*.

In the *Aryabhatiya* it is stated that "the circumference of a circle of diameter 20,000 units is approximately 62,832." Aryabhata recognises the value to be *asanna* i.e., a near approximation commenting on the word *asanna*, Nilakantha has given a fine disquisition on the meaning of incommensurability. "Why is it that only an approximate value of π is given? I shall explain. Because the exact value cannot be given. The unit that measures the diameter exactly will leave a remainder when it measures the circumference; just so the unit that measures the circumference exactly will leave a remainder when it measures the diameter. No (common) unit can be found which can measure both the diameter and circumference exactly. Even with samller and smaller units the situation remains the same; we will never come to a state of no remainder." [59].

Nilakantha gives detailed explanations and also brings in illustrations to elucidate the cryptic verses of Āryabhatiya. He has borrowed from earlier works and added some of his own.

The geometrical derivations of the following results presented in *Aryabhatiya-bhasya* are interesting:-

1. Area of a circle = half the circumference x half the diameter.

2. A side of a (regular) hexagon, inscribed in a circle = radius.

3. An approximation for the length of an arc of a circle:

$$\text{Arc} = Rx = \sqrt{(R\sin x)^2 + 4/3\,(R - R\cos x)^2}$$

4. Sum of an arithmetical progression.

5. The following results on summation:-

(a)

$$\sum_{r=1}^{n} \frac{r(r+1)}{2} = \frac{n(n+1)(n+2)}{6}$$

(b)

$$\sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}$$

(c)

$$\sum_{r=1}^{n} r^3 = \left[\frac{n(n+1)}{2}\right]^2$$

6. The second order difference of sines is proportional to the sine.

(Nilakantha's demonstrations of these results are given in Part II - Mathematical section of this book. Result 6 occurs in the derivation of the infinite series for sine and cosine; so it is not given separately).

Summing up. Nilakantha Somayaji was an eminent scholar as well as a great organiser and compiler. His *Tantra Samgraha* and *Āryabhatiya-bhāsya* are magnificent text-books. How much of the material in these works are his own is difficult to judge. Was he a creative mathematician? No definite answer is possible. There is one more feather to his cap; he was an excellent teacher. His students Jyeshta devan and Sankara Variyar bear testimony.

JYESTHA DEVAN

Jyestha devan (pronounced JYESH-THA-DEVAN) namputiri was a member of Parannottu (sanskritised as *Parakroda*) family in Alathur village, Ponnani taluk, Kerala. He has now been identified as the author of a popular treatise called the *Yukti-bhāsa* (pronounced YUKTI-BHAHSHA); it is also known as *Ganitanyāyasamgraham*. This work is written in Malayalam. It is devoted to a logical exposition of the mathematical and astronomical results extant then. The *Yukti-bhasa* holds an esteemed position as one of the most important works of an instructive type produced in the golden age.

Jyestha devan was a pupil of Damodaran Namputiri of Vatasreni, who earlier had taught Nilakantha. Jyestha devan was a younger contemporary and disciple of Nilakantha too. [47, 72; 74 & 89]

C.M. Whish records a tradition that the author of *Yukti-bhāsa* was also the author of a *Drkkarana*. This work is now available in a single manuscript. It is a metrical treatise on astronomy written in Malayalam. The last verse gives the date of completion in the words *kolambe barhisunau*; this, in the *Katapayadi*, denotes 783 of Malabar era or 1608 A.D. [74 & 109].

To have been a pupil of Damodaran, who died in 1510 A.D, Jyestha devan ought to have been born in 1500 A.D. Based on the evidence of *Drkkarana*, he must have lived up to 1608 A.D. We are led to surmise that Jyestha devan lived during the period 1500-1610 A.D.

It was the existence of a priestly class in Kerala that prompted the pursuit of art, literature and science to a large extent. There was a strong thread of continuity of tradition from teachers to students, and "knowledge" passed down from one generation to the next orally. Mathematics also followed the same tradition. One such teacher - student chain begins with Mādhavan of Sangamagrāmam and

proceeds successively to Paramesvaran Namputiri, his son Damodaran Namputiri, onto Nilakantha Somayaji and Jyestha devan.

Earlier works on mathematics, which are available now, give only the final results and "do so", "do such" instructions; explanations, derivations etc. were "taught" and "not recorded". Demonstrations of the results enunciated by Mādhavan must have reached Jyestha devan by word of mouth transmitted along the teahcer-student chain mentioned above.

The *Yukti-bhāsa* is comprehensive treatise comprising two parts. The first part is devoted to logical demonstration of mathematical results and the second part to astronomical topics. First five chapters of part one deal with the structure of integers, arithmetical operations, simple identities, solution of simultaneous equations in two unknowns, fractions, rule of three (proportion) and indeterminate equations. Most of these material can be found in earlier works. It is a recapitulation of previously known results to make the work self-sufficient. Sixth and seventh chapters contain proofs and derivations for some of the remarkable results mentioned above. (see Mādhavan). The following rapidly converging series for π have also been derived:-

(i) $$\frac{\pi}{16} = \frac{1}{1^5 + 4.1} - \frac{1}{3^5 + 4.3} + \frac{1}{5^5 + 4.5} - \dots\dots\dots$$

(ii) $$\pi = 3 + \frac{4}{3^3 - 3} - \frac{4}{5^3 - 5} + \frac{4}{7^3 - 7} - \dots\dots$$

(iii) $$\frac{\pi}{8} = \frac{1}{2^2 - 1} + \frac{1}{6^2 - 1} + \frac{1}{10^2 - 1} + \dots\dots\dots$$

(iv) $$\frac{\Pi}{8} = \frac{1}{2}\left[\frac{1}{4^2 - 1} + \frac{1}{8^2 - 1} + \frac{1}{12^2 - 1} + \text{..........}\right]$$

(v) $$\Pi \sim 2 + \frac{4}{2^2 - 1} - \frac{4}{4^2 - 1} + \text{.....} + \frac{4}{(2n)^2 - 1} \frac{-}{+} \frac{4}{2[(2n + 1)^2 + 2]}$$

The last chapter of part I concludes with discussion of a few results, not so profound as the foregoing, but each interesting in its own way:-

Result I.

$$\sin^2 x = x - \frac{x^4}{(2^2 - 2/2)} + \frac{x^6}{(2^2 - 2/2)(3^2 - 3/2)} - \text{......................}$$

$$+ (-1)^{n-1} \frac{x^{2n}}{(2^2 - 2/2)(3^2 - 3/2)(4^2 - 4/2) \text{.................} (n^2 - n/2)}$$

$$+ \text{.......................}$$

The pattern of factors in the denominators is lovely.

Result II:

sin (x ± y) = sin x, cos y ± cos x. sin y

The result is not new, as mentioned. The *Yukti - bhāsa* gives four methods of deriving this result.

Result III:-

If x, y, z are the three diagonals, and A, the area of a cyclic quadrilateral of sides a, b, c and d, then

$$x = \sqrt{\frac{(ab + cd)(ac + bd)}{(ad + bc)}} \qquad y = \sqrt{\frac{(ac + bd)(ad + bc)}{(ab + cd)}}$$

$$z = \sqrt{\frac{(ab + cd)(ad + bc)}{(ac + bd)}}$$

and $A = \sqrt{(s-a)(s-b)(s-c)(s-d)}$ Where $S = 1/2\,(a + b + c + d)$

Result IV:-

Surface area of a sphere of radius R

= circumference of a great circle x diameter

= $4\,\pi R^2$

and

Volume of the sphere = [surface area of the sphere x diameter] / 6

= $4/3\,\pi R^3$

(Demonstrations and further details of these results are given in part II - Mathematical section of this book.)

Sankara Variyar, another student of Nilakantha Somayaji, has done the same type of work. His turn is the next.

SANKARA VĀRIYAR

Among the prominent of works on mathematics and astronomy, Sankaran (pronouned SHAN-KARAN) deserves special mention. He was, by birth, a *paraśava* (called vāriyar in Malayalam) and by profession, a functionary of the temple at *Srihutāsa.*

Srihutāsa is a sanskritization of the Malayalam expression *Tr-k-kutaveli. Tr* is the equivalent of Sri and *kutaveli (kotuveli)* is the Malayalam name of a medical herb *citraka.* The names of the herb are synonyms of the word *hutās* a (fire).

The prime preceptor of Sankaran was Nilakantha Somayaji. Sankaran mentions this in several of his works. The student refers to the teacher as *sarva-vid* i.e. knower of all *sāstras.* Sankaran also says that Damodaran, Nilakantha's teacher was his teacher too. Sankaran had another teacher in Narayana, who was an *Azhvanceri Tamprakkal,* the (hereditary) religious head of namputiris. Narayana, besides being a teacher, was also a patron and benefactor of Sankaran. Sankaran describes Narayana as an erudite scholar and an abode of good qualities and himself as a favourite disciple of the revered Narayana. Sankaran has expressed his high esteem and regard for ' a brahmin of *Parakroda* family' - i.e., Jyestah devan [79 & 89].

The only original work of Sankaran known at present is the *Karanasāra,* an astronomical manual, written in the 1550's. The other extant works of Sankaran are all commentaries: the *Pancabodha-vyakhya* - a commentary written in 1529, on the *Pancabodha,* an anonymous work; the *Kriyākramakari* - an extensive commentary on the *Lilavati* of Bhaskara II; the *Yukti-dipika,* an elaborate commentary and the *Laghu - vivrti,* a concise one on the *Tantra Samgraha.* The *Laghu - vivrti* was his last work. [44, vol. II; 74 & 79].

Based on the dates of composition of his works, we can surmise that Sankaran lived between c. 1500 and c. 1560.

The *Lilavati* is one of the important non-Keralese works on mathematics, very popular among students in Kerala. As many as twelve Keralese commentaries on the *Lilavati* have been identified, five in Sanskrit and rest in Malayalam. Among these Sankaran's *Kriyākramakari* is the most important, both on account of the extensive exposition of the basic text and the analytic rationale it supplies. Sankaran could not finish this work; his commentary covers only the first 199 verses. One Narayanan namputiri (c. 1540 - 1610 A.D.) completed it.

There is a lot of overlapping in the material contained in the *yukti-Bhāsa* on the one hand and the *Yukti-dipika* and the *Kriyākramakari* on the other. Both Jyestha devan and Sankaran received instructions in *Jyotis'-sāstra,* perhaps together from Nilakantha; both studied the same works of earlier savants; they also held discussions on the topics they were studying. Sankaran acknowledges his indebtedness to Jyestha devan. Thus they drew extracts from common sources and held the same interpretations of the extracts. No wonder their works quote identical verses and show similarity in the exposition of certain topics. The three works mentioned above contain Mādhavan's verses enunciating mathematical results; demonstrations of various results too.

We have listed several results for which demonstrations are given in the *Yukti-bhāsa*. Many of them are derived in the *Kriyākramakari* too.

The golden age of Keralese mathematics comes to an end by 1610 A.D. Thereafter we come across 'technicians' not innovators. Only two mathematical treatises one, a selected compilation and the other a digest of mathematical results, have come to our knowledge. They are the *Karana Paddhati* and the *Sadratnamāla*.

59

PUTUMANA SOMAYAJI

The *Karana Pàddhati* was an important and widely used astronomical manual in Kerala. It gained popularity in the neighbouring areas (i.e. the present Tamilnadu and Andhra Pradesh) too. This is attested by the occurrence of its manuscripts in those regions. Two old commentaries, in Malayalam, are available. P.K. Koru has brought out a Malayalam edition of the work with his detailed and scholarly notes. Two Tamil and one Sanskrit commentaries are also known to exist. In spite of the high esteem and wide popularity won by the work, its author's name remains unknown. It remains as Putumana Somayaji i.e., a member of the Putumana family. *Nutanagrha* is the Sanskrit equivalent of the Malayalam Putumana. A plausible explanation for this is the following; to refer to a member of a family by the name of the family-house (with 'senior', 'junior' etc. added whenever necessary) has been a trait with certain sects in Kerala. Naturally a member of the Putumana family is referred to and even addressed as *'Putumana'* by his contemporaries. This practice has not died out even now. Because of this 'habit', names of individuals faded out and fell into oblivion. This is what might have happened in the case of the author of the *Karana Paddhati.*

The concluding verse of the *Karana Paddhati* explicitly mentions that the author was a native of *Shivapuram,* a *yajvav* (one who has performed *yajna* or sacrifice). Further the statement ganitametad *samyak in* the last verse is considered as the date, in the *Katapayadi,* of completion of the work. This number 1,765,653 reckoned in the *Kali* era, corresponds to a date in 1732 A.D.

Where is Shivapuram? Some hold the view that it is the present Trichur (Tr-s-Shivapuram). Except for Shivapuram occurring in the place name, odds are against this opinion. Namputiris of this locality are not privileged to perform soma sacrifices due either to Parasurama's curse or a royal degradation. Hence a native of this

place cannot be a somayaji. There is a neighbouring place called Covvaram (which is also sanskritized as Shivapuram), where even today there stands a house named Putumana. Putumana is from ancient days an important family of Covvaram; it is the first among a special group of eight house-holders known as *Pakazhiyam Astagrahakkars.* Further it is known as a house of traditional astronomers. They follow the *Asvalāyana sutra;* this is in conformity with our author's *Bahvrca-prayascitta,* which exemplifies the rites for the *Asvalāyanas.* [43, 44, vol. II; 47; 55 & 56].

The *Karana Paddhati* is a work in 213 verses packed in ten chapters. Starting from fundamentals it displays several formulae and tables which form the bases of *karana* texts or manuals. It deals only with selected topics. Besides the *Karana Paddhati,* Putumana Somayaji has to his credit several works such as the *Nyāya-ratna,* the *Venvārohastaka,* the *Pancabodha,* the *Grahana-ganita,* the *Jatakādesa,* the *Mānasa ganita,* the *Bahvrca-prāyascitta,* the *Āsauca* etc. [43; 44, vol. II; 47 & 74].

Almost all the mathematical results occuring in the *Karana Paddhati* are contained in the *Yukti-bhāsa,* the *Kriyākramakari* etc. Here is a result not seen in the afore-said works.

$$\frac{\pi}{6} = \frac{1}{2} + \frac{1}{(2.2^2-1)^2-2^2} + \frac{1}{(2.4^2-1)^2-4^2} + \frac{1}{(2.6^2-1)^2-6^2} + \dots$$

(This result has been derived along with similar results in Part II - Mathematical Section of this book.)

Clear evidences for the years of birth and death of Putumana Somayaji are not available. It is known that the *Karana Paddhati* was completed in 1732 A.D. Based on internal evidences available in the various other works of the author, K.V. Sarma assigns him the period c. 1660 - 1740, while K.K. Raja gives it as c. 1700 - 1760. [43; 47 & 74].

SANKARA VARMA

Sankara Varma was a member - more precisely, the younger brother of both king Udaya Varma and the heir - apparent Rama Varma - of the royal family of *Katattanat* in northern Kerala. He was born in 1800 A.D. For a good part of his early life, he did not get any education. Though he started on his studies, a little late, he made good ground in literature and *Jyotis'-sastra* in a short period. Later he came to be known as a talented poet, a gifted astrologer and an acute mathematical astronomer. Throughout his life he remained a staunch devotee of his family-deity viz., Goddess *Bhagavati* of *Lokanār kāv* (also called *Lokāmba*)

[44, vol. III ; 47 & 63]

Sankara Varma found a patron in Swati Tirunal maharajah, the king of Travancore (the southern part of Kerala), whom he visited in 1829. According to an anecdote Sankara Varma examined the horoscope of the king and predicted some future events in his life including the date of his demise. Further he told the king that his (Sankara Varma's) death would occur on a specific date in 1838. Sankara Varma passed away on the date predicted earlier. The king, on hearing the news was very much upset. [44, vol. III].

C.M. Whish knew Sankara Varma well. He considered Sankara Varma 'a very intelligent man and an acute mathematician. '[108, p. 521].

The *Sadratnamāla* is, perhaps, the only work of mathematics and astronomy written by Sankara Varma. The author acknowledges that his elder brother Prince Rama Varma was the driving force behind this venture. The work concludes with the chronogram *lokāmbasiddhasevye*. The chronogram, in the *Katapayadi*, is the number 1,797,313. This, in the *Kali* era, corresponds to a date in 1823 A.D. [44, vol. III; 47 & 74].

The *Sadratnamala* contains 211 verses arranged in six chapters. *Sad* means good, *ratna* precious stone, and *mala* garland; it is a garland of precious stones or a collection of valuable gems. The author makes no claim to originality anywhere and he has not provided any demonstrations of the results. The work is just a handbook of mathematical and astronomical results. The *Sadratnamāla* was composed in Sanskrit verse; the author then wrote a commentary in Malayalam. Unfortunately he was not able to complete the commentary - the last chapter remains unfinished.

The *Saḍratnamāla* gives the value of π correct to 17 decimal places.

A circle of diameter a *parardham* or 10^{17} units, has *bhadrambudhi siddha janmaganitasradhasmayat bhupagih* as its circumference; in the *Katapayadi* this is 314,159,265,358,979,324. [63 & 109].

The results occurring in the *Sadratnamala* can be traced back to earlier works but the exposition is new. They are in the author's own words not quotations. This shows that Sankara Varma had a good grasp of *Jyotis'-sāstra.*

To sum up. The golden age of Keralēse mathematics had a brilliant start, followed by a lustrous period; a dim twilight appearing as an aftermath.

PART - II
MATHEMATICAL SECTION

SOME CONCEPTS and THEIR NOMENCLATURE

We collect here some concepts and their nomenclature as found in the works of Indian mathematicians.

On account of the shape, an arc of a circle (eg. arc ACB in Figure - i) is called a *capa* or *dhanus* (Sanskrit words for the bow). The line segment AMB which looks like a bow-string is called *samasta jya* (full chord) and the line segment MA (or MB) is called *ardhajya* or *jya-ardha* (half chord). In their study of the circle the *ardha-jya* occurred very often, almost to the exclusion of *samasta-jya*. So the Indian mathematicians abbreviated *ardha-jya* to simply *jya*. [61 & 98].

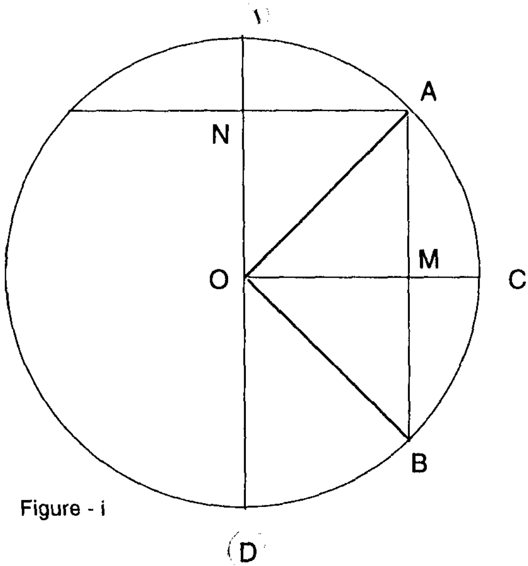


Figure - i

They had an alternate term for the line segment MA, viz., *bhujajya*. The line segment OM (or NA) was called *koti-jya*. When there is no room for confusion *bhuja-jya* is called simply *jya*. The term *jyas* denote the pair *bhuja-jya* and *koti-jya*.

It will be interesting to observe how the word *jya* degenerated into the word sine. An alternate word for *jya* viz. *jiva* became *jiba* at the hands of the Arabs. There is an Arabic word *jaib*, meaning "bay" or "inlet". Later there was some confusion between the words *jiba* and *jaib* (perhaps due to the omission of vowels) and the word was

translated into sinus, the Latin term for "bay" or "inlet". Hence came the word sine, providing an extreme example of a mathematical term which is completely bereft of its etymological meaning. [7; 102].

The *jya*, it must be noted was regarded as a length, being one half of the chord standing on double the arc.

Further they called the line segment MC, the *bhujā-saram* (pronounced BHUJAH-SHARAM) of arc CA and the line segment ND, the *koti-saram* of arc CA. Note that the *bhujā-saram* of arc CA = MC = OC - OM = Radius - *koti - jya* of arc CA and the *koti - saram* of arc CA = ND = OD - ON = OD - MA = Radius - *bhuja - jya* of arc CA.

The definition of *koti-saram*, given above, occurs in the *Yukti Bhāsa*. *Koti-saram* is used anywhere in the text and *saram* is used in place of bhujā-saram. The *Tantra Samgraha* and the *Karana Paddhati* use the terms *saram* or *utkrama-jya* denote *bhujā-saram*. [32; 58 & 89].

The treatises mentioned above have adopted the following convention; - whenever the terms *jya* and *saram* occur without any prefix or suffix they denote *bhujā-jya* and *bhuja-saram*, respectively.

***

Indian books on astronomy show two modes of division of the circumference of a circle into smaller parts. One is as follows:-

circumference	=	12 rāsis,
1 rasi	=	30 bhāgas or amsas
1 bhaga	=	60 kalas or liptas
1 kala	=	60 vikalas.

A second mode of division is as follows:-

circumference	=	21,600 (= 360 x 60) ilis

1 ili	=	60 vilis;
1 vili	=	60 talparas and
1 talpara	=	60 pratalparas,

In terms of ili, the radius of a circle is 3437 ilis, 44 vilis, 48 talparas and 22 pratalparas.

Radius is easily remembered, in the *katapayadi*, as

2 2 8 4 4 4 7 3 4 3

Srestho devo visvasthali bhrgu.

This number written in the reverse order is 3437444822. Pratalparas, talparas and vilis cannot exceed 59 and so cannot have more than two digits each. Hence the understanding is that the ten's and unit's places together denote the pratalparas, the thousand's and hundred's places together the talparas, the hundred thousand's and ten thousand's places together the vilis and remaining digits the ilis. This can also be written as 3437,44,48,22 where the commas separate ilis, vilis, talparas and pratalpara.

The two modes of division are essentially the same; only the terminologies are different. Ili and vili are the same as kala and vikala respectively.

Note that all these measures are linear; 1 bhāga will correspond to 1 degree, in the sense that 1 bhāga of arc will subtend an angle of one degree at the centre of the circle. Similarly 1 kala (ili) will correspond to 1 minute and 1 vikala (vili) to 1 second,

The radius, usually, is reckoned as 3437 ilis, 44 vilis and 48 talparas. This will correspond to 57 degrees, 17 minutes and 44.8 seconds i.e., a radian (of modern trigonometry).

* * *

We have given the nomenclature used by the Indian mathematicians, just to acquaint the reader with those terms. We refrain from using them in the sequel so as to avoid any confusion.

Let us *examine the method used by the Indian mathematicians* to study the circle and related topics. They divided the circumference of a circle into 21,600 (or 360 x 60) or 360 or 96 equal parts whichever was convenient. They proceeded with one of the equal parts as a unit. The unit was always an arc-length. We shall discuss the division into 360 equal parts; other cases can be treated similarly.

Let us look at the division of the circumference of a circle into 360 equal parts from the modern point of view. A unit of arc-length will *subtend an angle of magnitude one degree at the center of the circle;* the arc-length is $2\pi R / 360$ or $\pi R / 180$ where R is the radius of the circle. If R = 1, the unit, mentioned above, will be ($\pi / 180$), which is equivalent to the radian measure of one degree; so the length of any arc on a unit circle is equal to the radian measure of the angle associated with the arc.

Let us *switch back to the circle of radius R and see what* happens. Figure-ii, shows two concentric circles with the common center 0 and radii 1 and R. Let the arc C 'A' on the unit circle = x units. OA' produced meets the circle of radius R at A; AM and A'M' are perpendiculars to OC. Now

$$\frac{\text{arc } CA}{R} = \frac{\text{arc } C'A'}{1}$$

So arc CA = R. arc c'A' = R.x

The two triangles OAM and OA'M' are similar; so

$$\frac{AM}{A'M'} = \frac{OM}{OM'} = \frac{OA}{OA'} = \frac{R}{1}$$

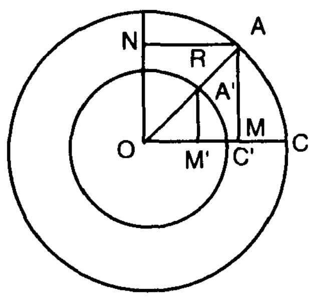


Figure ii

Hence AM = R.A'M' = R. sin x and OM = R.OM' = R.cosx

In short, the transition from the unit circle to a circle of radius R magnifies the arc-length, sine and cosine R times.

***

Throughout this book we adopt the following conventions:-

We take the radius of the circle = R and the arc CA to correspond to x units. Then, as has been explained with the help of Figure - ii

arc AC = R.x

AM = the sine of the arc AC = R. sinx (not sin R.x)

OM = NA = the cosine of the arc AC = R.cos x and

MC = the versine of the arc AC = R. (1 - cos x).

Note that when the arc corresponds to x units (of arc-length), it subtends x units (in the angle measure) at the center of the circle. If the unit of arc-length is (1/360) of the circumference of the circle (i.e. π.R / 180), it subtends an angle of 1 degree at the center; again if the unit of arc-length is the radius of the circle, it subtends an angle of 1 radian at the center.

BHASKARA'S Approximation for sinx

Bhaskara I has given the ensuing numerical approximation for sin x, where the arc-lengths are expressed in terms of a unit = (1/360) of the circumference of the circle. Recall that this unit of arc-length = (π.R/180), corresponds to 1 degree (or '0') of angle-measure:-

sinx = [y (180 - y)] / [(1/4) 40,500 - y (180 - y)]

where y = x, 180 - x, x - 180 or 360 - x, according as the terminal point of the arc of length x units is in the first, second, third of fourth quadrant respectively. So the value of y is always less than 90, Ignoring the signs sinx = sin (180 - x) = sin (x - 180) = sin (360 - x).

METHOD I : In Figure - iii, let O be the center of a circle of radius R and PQ a diameter. Let the arc QM = x units Then MN, the perpendicular from M to PQ = R sinx, The area of the right triangle PMQ =

1/2 QM.MP; also 1/2 PQ.MN.

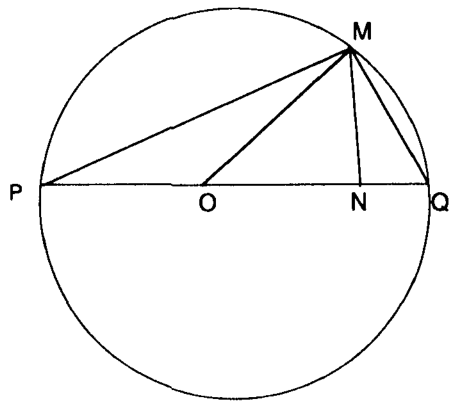
Hence 1 / MN = PQ / QM.MP

Any arc of a circle is longer than the chord joining its end-points; hence

1/MN > PQ / (arc QM) (arc MP)

Now let

Figure - iii



$$\frac{1}{MN} = \frac{A.PQ}{(\text{arc } QM)\,(\text{arc } MP)} + C$$

Where A anc C are constant to be determined.

Here arc QM = x units and arc MP = (180 - x) units, where the unit of arc-measurement = (π.R. / 180), which corresponds to an angle of 1 degree. So

$$\frac{1}{R\,.\,\sin x^{0}} = \frac{A.2R}{x\,.\,\frac{\pi R}{180} \times (180-x)\,\frac{\pi R}{180}} + C,$$

Let A = B. $(\pi/180)^2$,

then the above equation becomes

$$\frac{1}{\sin x^{0}} = \frac{2B}{x.(180\text{-}x)} + C.R, \text{ whence}$$

$$\sin x^{0} = \frac{[x.(180-x)]}{2B + x.(180-x).C.R.} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{I}$$

When x = 30, equation I reduces to
2B + 4500 C.R. = 9000 II

Again, when x = 90, equation I reduces to
2B + 8100 C.R. = 8100 III

The equations II & III give
2B = 40,500/4 and C.R. = - 1/4
Substituting these values in equation I, we get

$$\sin x^{o} = \frac{4x.(180-x)}{40{,}500 - x\,(180-x)} \text{. (Q.E.D.) [29]}$$

METHOD II: -

Let

$$\sin x^0 = \frac{a + bx + cx^2}{p + qx + rx^2}$$

where x is measured in the unit mentioned in the method I and a,b,c,p, q and r are multipliers (or constants) to be determined,

Here b & q include the factor (π / 180) and c & r the factor $(\pi / 180)^2$. Taking x = 0, we get a = 0.

Then x = 180 gives b + 180 c = 0; hence $c = \frac{-b}{180}$

Thus

$$\sin x^0 = \frac{bx\,(180 - x) / 180}{p + qx + rx^2}$$

Since sinx = sin (180 -x), we have

$$\frac{b.x.(180 - x) / 180}{p + qx + rx^2} = \frac{b.\,(180 - x)\,.x / 180}{p + q.\,(180 - x) + r.\,(180 - x)^2}$$

So $p + qx + rx^2 = p + q.\,(180 - x) + r.\,(180 - x)^2$,
i.e. q. (2x - 180) = r.180 (180 - 2x), giving

$$r = \frac{-q}{180}$$

Therefore

$$\sin x^0 = \frac{b.x.\,(180 - x)}{p.180 + q.x.\,(180 - x)} \qquad \text{......................IV}$$

when x = 30, equation IV reduces to

P + 25q = 50b V

Again, when x = 90, equation IV reduces to
2p + 90 q = 90 b VI

Equations V and VI, yield

$$p = \frac{225\,b}{4} \quad \text{and } q = \frac{-b}{4}$$

Substituting these values in equation IV, we have

$$\sin x^{0} = \frac{b.x.\,(180 - x)}{(225.180b\,/\,4) \;-\; (b.x.\,(180 - x)\,/\,4)}$$

$$= \frac{4.x.\,(180 - x)}{40{,}500 - x.\,(180 - x)} \qquad \text{(Q.E.D.)}$$

Both the methods detailed above, can be worked out, on identical lines, when the unit of arc-measure is a radius (which corresponds to a radian in angle-measure).

The result, then, is

$$\sin x = \frac{16.x.\,(\pi - x)}{5.\pi^2 - 4.x\,(\pi - x)}$$

When $x = \frac{\pi}{n}$, the result is

$$\sin\frac{\pi}{n} = \frac{16\,(n - 1)}{5.n^2 - 4\,(n - 1)}$$ This can be put in the more elegant form:-

$$\sin\frac{\pi}{n} = \frac{n^2 - (n - 2)^2}{n^2 + 1/4\,(n - 2)^2}$$

This elegant form is due to Ganesa, an Indian mathematician of the sixteenth century. [34].

MADHAVAN, the father of INFINITESIMAL ANALYSIS

Mādhavan of Sangamagramam (1340 - 1425 A.D.) has been identified as the world's first mathematician to enunciate the infinite series expansion for inverse tangent, sinx etc. Mādhavan's work containing these results remains to be unearthed; so we do not have any first hand information as to how he arrived at these results. We have to depend on the writings of later authors. Jyestha-devan's Yukti Bhāsa is one of the earliest treatises now available which contains demonstrations of Madhavan's results and some corollaries also. Here we present the material found in the Yukti Bhasa in as faithful a manner as possible.

What is the circumference, C, of a circle of diameter D? or what is the value of π i.e. the ratio of C to D? This question has been engaging the attention of mathematicians of several civilizations from very early times.

Early attempts

Let us recall some definitions and conventions;

(1) A polygon is said to be CIRCUMSCRIBED about a circle if

(a) the circle is enclosed in the polygon and

(b) each side of the polygon is a tangent to the circle.

(2) A polygon is said to be INSCRIBED in a circle if

(a) the polygon is enclosed in the circle and

(b) all the vertices of the polygon are points on the circle.

(3) A polygon P_2 is said to be INSCRIBED in a polgon P_1 if

(a) the polygon P_2 is enclosed in P_1 and (b) all the vertices of P_2 are points of P_1 (the sides of P_2 & P_1 may or may not overlap).

(4) A polygon is called REGULAR if all its sides are equal.

METHOD I : We get approximations to the circumference of a circle by means of a sequence of regular polygons, each inscribed in the preceding one and circumscribing the circle. We join the vertices of a polygon to the center of the circle. At the points where these joins (whose segments look like the spokes of a circular wheel) interesect the circle, perpendiculars are drawn to the joins. These perpendiculars are produced both ways so as to meet two adjacent sides of the polygon. Segments of these perpendiculars, which are tangents to the circle, together with segments (of the sides of the polygon) on which the points of contact with the circle lie, form the subsequent polygon in the sequence. Not that the new polygon is inscribed in the polygon we started with, and circumscribes the circle. Further the number of sides has doubled. the length of a side of any polygon in the sequence can be calculated from that of the preceding one, using simple properties of triangles.

METHOD II : In another method we start with a regular hexagon inscribed in the circle. The six point at which the perpendicular bisectors of the sides of the hexagon interesect the circle, are the additional vertices of the subsequent polygon (of 12 sides). Continuing this process we can get regular polygons of 24; 48; 96; 192; 384......... sides, all inscribed in the circle. The length of a side of any polygon in the sequence can be calculated from that of the preceding one by repeated use of Pythagoras theorem.

Archimedes (287-212 B.C.) of Syracuse combined the two methods. Starting with inscribed and circumscribed regular hexagons he went on doubling the numbers of sides till he reached ninety-six, in each case. Computation of their perimeters led him to the result:

$$3\frac{10}{71} < \frac{C}{D} < 3\frac{10}{70}$$

This result is given in proposition 3 of the Archimedean treatise On the measurement of the circle - probably incomplete, as it has

come down to us. [7; 100 - I]

Aryabhata in his *Aryabhatiya* (499 A.D.) has stated "that the circumference of a circle of diameter 20,000 is 62,832 approximately." We can get this result by method I; start with a square (a regular polygon of four sides) and construct a sequence of regular polygons of sides 8;16; 32; 64; 128; 256 .... respectively. The circumscribed regular polygon of 256 sides gives the result. Here π = 3.1416.

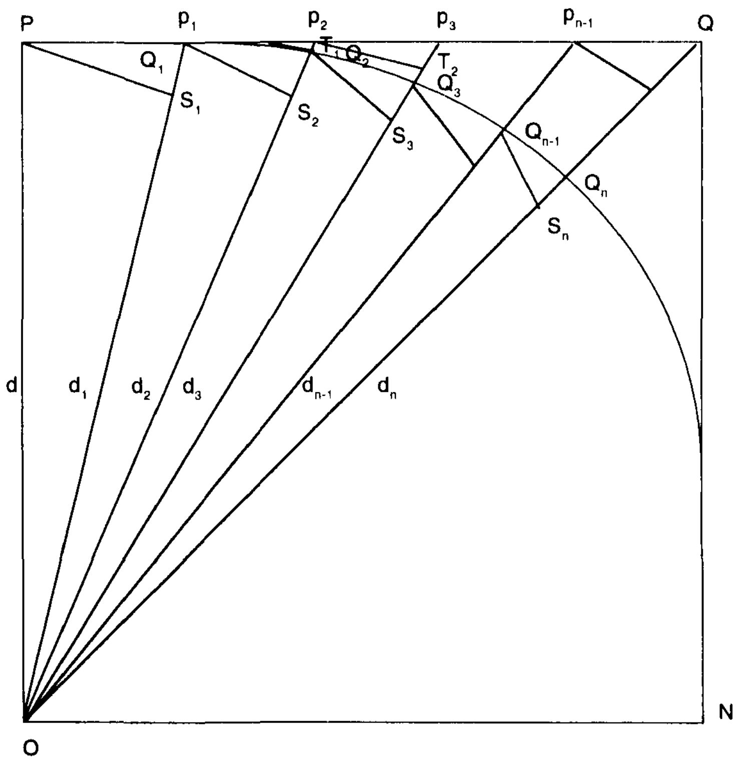
Bhaskara II (1114 - c. 1185) in his *Līlavati*, according to the commentator Ganesa (16th century), derived the result "that the circumference of a circle of diameter 1250 is 3927", using a regular polygon of 384 sides, inscribed in the circle. (The perimeter of this regular inscribed polygon is 3926.9376 correct to 4 decimal places which gives π the value 3.14155).

Madhavan's infinite series for (π/4) and inverse tangent

Next we come to a method of getting the ratio of the circumference of a circle to its diameter as an infinite series. The argument used is one of "induction" but not as it is understood today. Professor Andre'Weil has made it clear in the following words; "In those days (1630s), and even much later, the word "induction" was used in an altogether different sense; it indicated no more than what is now understood by "conjecture". To verify "by induction" a statement P(n), depending upon an integer n, meant merely to verify it for some (and sometimes very few) initial values of n. Of course it could happen that the procedure used, say, in order to verify P(4) was to derive it from P(3) in such a manner as to make it obvious that the same argument would serve to derive P(n) from P(n-1). In that case, which occurs not seldom in Euclid (e.g. in Eucl. I. 45, VII, 14, VIII, 6, to give a few random examples) a mathematician has the right to regard such a proof as conclusive, even though it is not couched in the conventional terms which would be used for it today." [108]. This remark, about the situation in the 17th century Europe, is true of that in the 16th century India also.

We start with a circle. A square of side equal to the diameter of the circle is drawn circumscribing it. Note that the circle touches the square at the midpoints of its sides. The points of contact on the opposite sides of the square are joined. Only one quadrant of the diagram is shown in Figure-iv. PQ and QN are halves of the adjacent sides of the square; 0, the center of the circle; OP, ON are the radii, R, of the circle. The line-segment PQ is divided into n equal parts by the points P_1, P_2............................P_{n-1}, P_n (=Q). The joins of these points to o interesect the circle at the points Q_1, Q_2................Q_{n-1}, Q_n respectively.

Figure - iv



Perpendicular to OP_1 is drawn from P to meet it at S_1; further the perpendiculars from P_{r-1} and Q_{r-1} to OP_r meet it at T_{r-1} and S_v respectively, where r = 2, 3 ....n.

Let us assume the length of each of the equal line-segments PP_1, P_1P_2, ..., $P_{n-1}Q$ to be equal to a. Further let us denote the lengths of OP, OP_1, Op_2, ...OQ by d_0, d_1, d_2, ... d_n respectively. Note that $d_0 = R$ and $dn = \sqrt{2}.R$ where R is the radius of the circle.

We proceed to calculate the lengths of the line-segments PS_1, Q_1S_2, Q_2S_3 .... & Q_{n-1} S_n. Consider the right triangles OPP_1 and PS_1P_1. They are similar, since two sides of one triangle overlap with and two are perpendicular to two each of the other triangle.

Hence

$$\frac{PS_1}{PP_1} = \frac{OP}{OP_1} \quad \frac{d_0}{d_1} \quad \text{i.e.} \quad PS_1 = \frac{a.d._0}{d_1}$$

Again the right triangle OPP_2 and $P_1T_1P_2$ are similar and so

$$\frac{P_1T_1}{P_1P_2} = \frac{OP}{OP_2} = \frac{d_0}{d_2} \quad \text{i.e.} \quad P_1T_1 = \frac{a.d._0}{d_2}$$

Since the right triangles OP_1T_1 and OQ_1S_2 are similar, we have

$$\frac{OQ_1}{OP_1} = \frac{Q_1S_2}{P_1T_1} \text{ or } Q_1S_2 = \frac{P_1T_1.OQ_1}{OP_1} = \frac{a.d_0}{d_2} \times \frac{d_0}{d_1} = \frac{a.d_0^2}{d_1.d_2}$$

Further the right triangles OPP_3 and $P_2T_2P_3$ are similar; so

$$\frac{P_2T_2}{P_2P_3} = \frac{OP}{OP_3} \text{ or } P_2T_2 = \frac{P_2P_3.OP}{OP_3} = \frac{a.d_0}{d_3}$$

The right triangles OP_2T_2 and OQ_2S_3 being similar, we have

$$\frac{Q_2S_3}{P_2T_2} = \frac{OQ_2}{OP_2} \text{ or } Q_2S_3 \frac{P_2T_2.OQ_2}{OP_2} = \frac{a x d_0}{d_3}. \frac{d_0}{d_2} = \frac{a.d_0^2}{d_2.d_3}$$

Continuing in this manner, we get

$$Q_{r-1} S_r = \frac{a.\ d_0^2}{d_{r-1}\ .\ d_r}$$

Here PS_1, Q_1S_2, ... $Q_{r-1}S_r$, ... are the respective sines of (i.e., half-chords of double) the arcs PQ_1, Q_1Q_2, ...Qr-1 Qr, ...

When the line-segments PP_1, P_1P_2, P_2P_3, ... are each very small, the corresponding arcs, PQ_1, Q_1Q_2, Q_2Q_3, ... of the circle will also be very small and then we can suppose the arcs PQ_1, Q_1Q_2, Q_2Q_3, ... to be respectively equal to their sines PS_1, Q_1S_2, Q_2S_3, ... Sum of the arcs $PQ_1 + Q_1Q_2 + Q_2Q_3 +$ .... $Q_{n-1}Q_n$ is an octant of the circle. Assuming that the circumference of the circle is C, we can write

$$c / 8 \sim \text{Sum of the sines} = PS_1 + Q_1S_2 + Q_2S_3 + \ldots\ldots + Q_{n-1}S_n$$

$$= \frac{a.R^2}{d_0.d_1} + \frac{a.R^2}{d_1,d_2} + \frac{a.R^2}{d_2,d_3} + \ldots\ldots\ldots + \frac{a.R^2}{d_{n-1}\ .\ d_n},$$

since $R = d_0$

If the number of equal parts into which PQ has been divided, i.e., n, is very large, any two adjacent d's namely d_{r-1} and d_r can be supposed to be equal. i.e., $d_r - d_{r-1} \rightarrow 0$.

Hence $(d_r - d_{r-1})^2 = d_r^2 + d_{r-1}^2 - 2d_r\ .\ d_{r-1} \rightarrow 0$

i.e., $d^2_{r-1} + d^2_r \longrightarrow 2\, d_{r-1} \cdot d_r$

or $d_{r-1} \cdot d_r \longrightarrow 1/2\, (d^2_{r-1} + d_r^2)$

Now

$$\frac{1}{d_{r-1} \cdot d_r} \longrightarrow \frac{2}{d^2_{r-1} + d^2_r} = \frac{2\,(d^2_{r-1} + d^2_r)}{(d^2_{r-1} + d^2_r)^2}$$

$$\longrightarrow \frac{2\,(d^2_{r-1} + d_r^2)}{4.\, d^2_{r-1} \cdot d^2_r} = \frac{1}{2}\left(\frac{1}{d^2_{r-1}} + \frac{1}{d^2_r}\right)$$

Applying this result to the equation (A), we get

$$\frac{1}{8}C \sim \frac{1}{2}\left(\frac{a.R^2}{d^2_0} + \frac{a.R^2}{d_1^2}\right) + \frac{1}{2}\left(\frac{a.R^2}{d_1^2} + \frac{a.R^2}{d_2^2}\right) + \ldots\ldots\ldots\ldots +$$

$$\ldots\ldots\ldots\ldots\ldots + \frac{1}{2}\left(\frac{a.R^2}{d^2_{n-1}} + \frac{a.R^2}{d_n^2}\right)$$

$$= \frac{1}{2}\left(\frac{a.R^2}{d^2_0} + \frac{a.R^2}{d_1^2} + \ldots\ldots \frac{a.R^2}{d^2_{n-1}}\right)$$

$$+ \frac{1}{2}\left(\frac{a.R^2}{d^2_1} + \frac{a.R^2}{d_2^2} + \ldots\ldots \frac{a.R^2}{d^2_n}\right)$$

Difference between the sums of the two sets in the brackets

$$\frac{aR^2}{d_0^{\,2}} - \frac{aR^2}{dn^2} = \frac{aR^2}{R^2} - \frac{aR^2}{2.R^2}$$

since $d_n = OQ = \sqrt{2}\,.\,R$

$$= a - \frac{a}{2} = \frac{a}{2}$$

Half the difference $= \dfrac{a}{4} \rightarrow 0$ with a.

Hence we can assume the two sums to be equal; so

$$\frac{1}{8}C \sim \frac{aR^2}{d_1^{\,2}} + \frac{aR^2}{d_2^{\,2}} + \ldots\ldots\ldots\ldots + \frac{aR^2}{d_r^{\,2}} + \ldots\ldots + \frac{aR^2}{d_n^{\,2}} \quad (B)$$

Now consider the expression

$$\frac{a.R^2}{R^2} - \frac{a.(ra)^2}{R^2} + \frac{a.(ra)^4}{R^2} - \ldots\ldots\ldots (-1)^{m-1}\frac{a.(ra)^{2m-2}}{R^{2m-2}} +$$

$$(-1)^m \frac{a.\,(ra)^{2m}}{R^{2m-2}.d^2_r} \qquad (C)$$

The last terms of the expression (C) can be simplified thus:

$$(-1)^{m-1}\frac{a.\,(ra)^{2m-2}}{R^{2m-4}}\left(\frac{1}{R^2} - \frac{(ra)^2}{R^2.d^2_r}\right)$$

$$= (-1)^{m-1} \frac{a.(ra)^{2m-2}}{R^{2m-4}} \left(\frac{d^2_r - (ra)^2}{R^2.d^2_r} \right)$$

$$+ (-1)^{m-1} \frac{a.(ra)^{2m-2}}{R^{2m-4}.d^2_r} \text{, since } d^2_r = R^2 + (ra)^2.$$

Repeating this process of combining the last two terms successively, the expression (C) finally reduces to :

$$\frac{a.R^2}{R^2} - \frac{a.(ra)^2}{d^2_r} = a\left(1 - \frac{(ra)^2}{d^2_r}\right) = a\frac{d^2_r - (ra)^2}{d^2_r} = a.\frac{R^2}{d^2_r}$$

which is the r-th term on the right hand side of (B).

In the *Yukthi-bhāsa* this result is derived for r = 1 and m = 1 & 2.

Each term of (B) can be replaced by an m - termed expression as in (C). Regrouping the terms by collecting the first terms of the expressions, the second terms and so on, we get

$$\frac{1}{8}C \sim n.a - \frac{a.a^2}{R^2}(1^2 + 2^2 + ...n^2) +$$

$$\frac{a.a^4}{R^4}(1^4 + 2^4 + ... + n^4) ....$$

$$.............. + (-1)^m \frac{a.a^{2m}}{R^{2m-2}} \left(\frac{1^{2m}}{d_1^2} + \frac{2^{2m}}{d_2^2} + ... \frac{n^{2m}}{d_n^2}\right) \text{(D)}$$

We take a diversion here, in order to re-collect some results already known to Mathematicians of earlier periods and to present a couple of results, established in the Yukti-bhāsa, as lemmas.

Āryabhata was familiar with the arithmetic series and its sum

$$1 + 2 + 3 + ... + n = \frac{n(n+1)}{2}$$

Nilakantha Somayaji (1444 - 1545 A.D.) had demonstrated the results (see Nilakantha's Geometrical demonstrations) : -

$$1^2 + 2^2 + 3^2 + \ldots\ldots\ldots\ldots n^2 = \frac{n.(n+1).(2n+1)}{6}$$

$$\text{and } 1^3 + 2^2 + 3^3 + \ldots\ldots\ldots\ldots\ldots\ldots n^3 = \frac{n^2.(n+1)^2}{4}$$

Lemma I : Let

$$S_0^{(p)} = 0 \text{ and } S_n^{(p)} = 1^p + 2^p + 3^p + \ldots\ldots\ldots\ldots + n^p \text{ for } n > 0;$$

$$(p = 1, 2, 3, \ldots\ldots\ldots\ldots)$$

then

$$n.S_n^{(p-1)} = S_n^{(p)} + S_1^{(p-1)} + S_2^{(p-1)} + \ldots\ldots\ldots\ldots + S_{n-1}^{(p-1)}$$

Proof : -

$$S_n^{(p-1)} = 1^{p-1} + 2^{p-1} + 3^{p-1} + \ldots\ldots + n^{p-1}$$

$$\text{So, } n.s_n^{(p-1)} = n.1^{p-1} + n.2^{p-1} + \ldots\ldots\ldots + n.(n-1)^{p-1} + n.n^{p-1}$$

$$= 1^p + 2^p + 3^p + \ldots\ldots(n-1)^p + n^p$$

$$+ (n-1).1^{p-1} + (n-2).2^{p-1} + + (n-3).3^{p-1} + \ldots\ldots + 1.(n-1)^{p-1}$$

$$= S_n^{(p)} + 1^{p-1} + 2^{p-1} + 3^{p-1} \ldots\ldots\ldots + (n-1)^{p-1}$$

$$+1^{p-1} + 2^{p-1} + 3^{p-1} + \dots\dots + (n-2)^{p-1}$$

— — — — — — — — — — — — — — — —

— — — — — — — — — — — — — — — —

$$+1^{p-1} + 2^{p-1}$$

$$+1^{p-1}$$

$$= S_n^{(p)} + S_{n-1}^{(p-1)} + S_{n-2}^{(p-1)} + \dots\dots\dots\dots\dots + S_2^{(p-1)} + S_1^{(p-1)}$$

Lemma II :

When n is very large

$$S_n^{(p)} = 1^p + 2^p + 3^p + \dots\dots\dots + n^p \sim \frac{n^{p+1}}{p+1}$$

Proof: Assume the result true for (p-1) i.e.,

$$S_n^{(p-1)} \sim \frac{n^p}{p} \quad \text{or} \quad S_{n-1}^{p-1} \sim \frac{n^p}{p}$$

then $S_{n-1}^{(p-1)} + S_{n-2}^{(p-1)} + \dots\dots\dots\dots\dots + S_1^{(p-1)} + S_0^{(p-1)} \sim \frac{1}{p} S_n^{(p)}$

This result is known as the Cauchy-Stolz limit theorem[1]. Now applying the result of lemma I, we get

$$n.s_n^{(p-1)} - S_n^{(p)} \sim \frac{1}{P} \cdot S_n^{(p)}$$

$$\text{i.e., } n.s_n^{(p-1)} \sim \frac{p+1}{p} S_n^{(p)}$$

(1) **The Cauchy-Stolz limit theorem:**

"If (E_n), (D_n) are two sequences, of which the latter, (D_n) tends to +infinitey, then

$$\lim_{n \to \infty} \frac{E_n}{D_n} = \lim_{n \to \infty} \frac{E_n - E_{n-1}}{D_n - D_{n-1}}$$

provided the latter limit exists."

makes this result immediate. [XII].

How this result was arrived at in the Yukti-bhāsa, which was written before 1610 A>D., leaves us amazed.

Using the assumption that $S_n^{(p-1)} \sim \frac{1}{p} n^p$, we get

$$\frac{1}{p} . n^{(p+1)} \sim \frac{p+1}{p} S_n^{(p)} \text{ or } S_n^{(p)} \sim \frac{1}{p+1} . n^{p+1}$$

Thus assuming the result true for (p-1), we have derived it for p.

We know that the result is true for the value 1, i.e.,

$$S_n^{(1)} = 1 + 2 + 3 + \ldots\ldots\ldots\ldots. + n = \frac{n(n+1)}{2} = \frac{n^2}{2} + \frac{n}{2} \sim \frac{n^2}{2}$$

So the result is true for p = 2 and hence for all positive integral values of p.

Lemmas I & II have been proved for the values p = 1, 2, 3, and 4 in the Yukti-bhāsa.

Application of lemma II to the equation (D), reduces it to:

$$\frac{1}{8}C \sim na - \frac{n^3 . a^3}{3.R^2} + \frac{n^5 . a^5.}{5.R^4} - \dots\dots\dots + (-1)^m \frac{a^{2m+1}}{R^{2m-2}} \sum_{r=1}^{n} \frac{r^{2m}}{d_r^2}$$

$$\dots\dots\dots\dots\dots\dots\dots\dots \text{(E)}$$

We know that $d_r > R$, for each of the values r = 1, 2, 3, ...n.

Hence

$$\frac{a^{2m+1}}{R^{2m-2}} \sum_{r=1}^{n} \frac{r^{2m}}{d_r^2} < \frac{a^{2m+1}}{R^{2m-2}} \sum_{r=1}^{n} \frac{r^{2m}}{R^2} = \frac{a^{2m+1}}{R^{2m}} \sum_{r=1}^{n} r^{2m}$$

$$\sim \frac{a^{2m+1}}{R^{2m}} . \frac{n^{2m+1}}{2m+1} = \frac{R}{2m+1},$$

Since n.a = R. and

$$\frac{R}{2m+1} \rightarrow 0 \text{ when m tends to infinity}$$

Hence $\frac{1}{8}C = R - \frac{R}{3} + \frac{R}{5} - \frac{R}{7} + \dots\dots\dots\dots\dots\dots$

or $\frac{1}{4}C = D - \frac{D}{3} + \frac{D}{5} - \frac{D}{7} + \dots\dots\dots\dots$

Where D is the diameter of the circle

We start with the Figure - v, of a quadrant of a circle enclosed in the two mutually perpendicular radii OP, OS and the two halves (PQ, QS) of the two adjacent sides of the square circumscribing the circle.

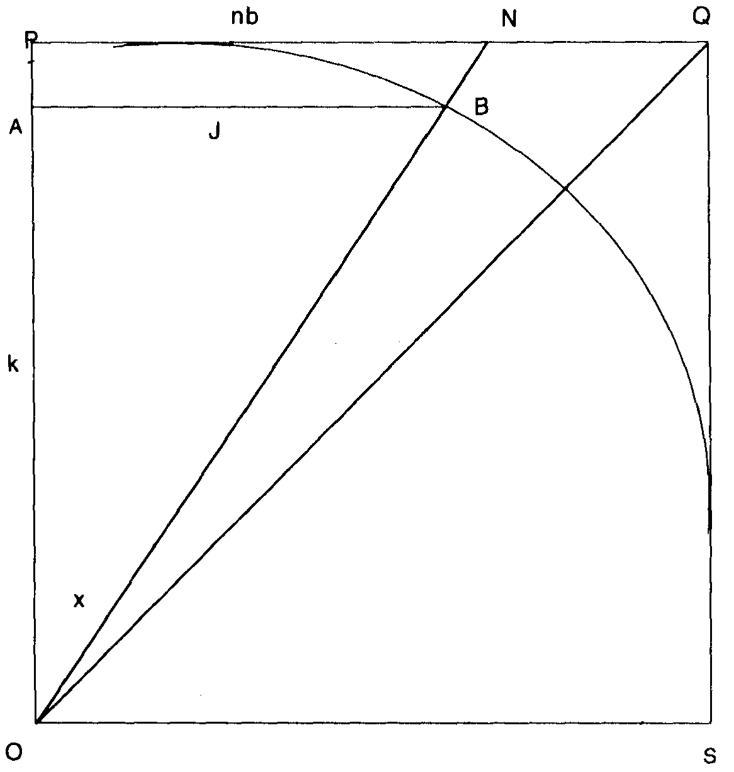


Figure - v

The points O and Q are joined. Let B be any point on the circle such that the arc PB is less than an octant of the circle. The point O is joined to B and produced further to meet the line-segment PQ at N. The line-segment PN is divided into n equal parts. We assume that each of the equal parts - b and so PN = n.b.

We derived the equation (E) using the Fig. iv; just so Fig. - v gives

$$\text{arc PB} = nb - \frac{(nb)^3}{3.R^2} + \frac{(nb)^5}{5.R^4} - \frac{(nb)^7}{7.R^6} + \ldots\ldots$$

(The absolute value of the term here, which corresponds to the m-th term in (E), will be less than the absolute value of the m-th term in (E), because $b < a$; hence convergence of the series follows.)

A perpendicular from B is dropped on OP to meet it at A.

Now, the right triangles OAB and OPN are similar. So

$$\frac{PN}{AB} = \frac{OP}{OA} \quad ; \text{ hence } PN = \frac{AB.OP}{OA} \quad \text{or} \quad nb = \frac{J.R}{k},$$

Where $j = AB$, $k = OA$ and $OP = R$, the radius of the circle.

Replacing nb by the value obtained above, we get

$$\text{arc PB} = \frac{(J.R.)}{k} - \frac{(J.R.)^3}{3.k^3.R^2} + \frac{(J.R.)^5}{5.k^5.R^4} - \frac{(J.R.)^7}{7.k^7.R^6} + \ldots\ldots\ldots\ldots\ldots$$

$$= R.\ \frac{j}{k} - \frac{R}{3} . \frac{J^3}{K^3} + \frac{R}{5} . \frac{J^5}{K^5} - \frac{R}{7} . \frac{J^7}{K^7} + \ldots\ldots(F)$$

This is the form in which Mādhavan enunciated the result.

If we take arc $PB = R.x$, $j = R.\sin x$ and $k = R.\cos x$

we get

$$R.x = R.\frac{\sin x}{\cos x} - \frac{R}{3}.\frac{(R.\sin x)^3}{(R.\cos x)^3} + \frac{R}{5}\frac{(R.\sin x)^5}{(R.\cos x)^5} - \ldots\ldots\ldots\ldots$$

or

$$x = \tan x - \frac{\tan^3 x}{3} + \frac{\tan^5 x}{5} - \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

which is now known as the Gregory series.

In (F), if we take arc PB = 1 / 12 C, then the corresponding,

$$j = \frac{R}{2} = \frac{D}{4} \text{ and } k^2 = R^2 - \frac{1}{4}.R^2;$$

so $k = \frac{\sqrt{3}}{2}.R = \frac{\sqrt{3}}{4}.D$, where 2R = D, the diameter of the circle ; then $\frac{J.R}{k} = \frac{1}{12}.D$,

leading to result:

$$C = \sqrt{12.D^2} - \frac{\sqrt{12.D^2}}{3.3} + \frac{\sqrt{12.D^2}}{3^2.5} - \frac{\sqrt{12.D^2}}{3^3.7} + \ldots$$

Some rapidly converging series for C

The Yukti Bhāsa gives the following results and their derivations. It is not known who discovered them; Mādhavan or somebody else.

Starting with the result:

$$C = 4D - \frac{4D}{3} + \frac{4D}{5} - \frac{4D}{7} + \cdots\cdots$$

several rapidly convergent series for C have been derived.

(i) We rewrite the above series as

$$C = 4D - \frac{4D}{2.1 + 2 + \frac{4}{2.1+2}} + \left\{ \frac{4D}{2.1 + 2 + \frac{4}{2.1+2}} - \frac{4D}{3} + \frac{4D}{2.3+2+\frac{4}{2.3+2}} \right\}$$

$$- \left\{ \frac{4D}{2.3 + 2 + \frac{4}{2.3+2}} - \frac{4D}{5} + \frac{4D}{2.5+2+\frac{4}{2.5+2}} \right\} \cdots\cdots$$

— — — — — — — — — — — — — — — — — — —

— — — — — — — — — — — — — — — — — — —

$$\pm \left\{ \frac{4D}{2(2n-1) + 2 + \frac{4}{2(2n-1)+2}} - \frac{4D}{2n+1} \quad \frac{4D}{2(2n+1)+2+\frac{4}{2(2n+1)+2}} \right\}$$

— — — — — — — — — — — — — — — — — — —

— — — — — — — — — — — — — — — — — — —

Now consider the set of terms

$$\frac{4D}{2\,(2n-1) + 2 + \dfrac{4}{2\,(2n-1)+2}} - \frac{4D}{2n+1} + \frac{4D}{2(2n+1)+2+\dfrac{4}{2\,(2n+1)+2}}$$

$$= \frac{4D}{4n + \dfrac{1}{n}} - \frac{4D}{2n+1} + \frac{4D}{4(n+1) + \dfrac{1}{(n+1)}}$$

$$= 4D\left\{\frac{n}{4n^2+1} - \frac{1}{2n+1} + \frac{n+1}{4(n+1)^2+1}\right\}$$

Which on-simplification reduces to

$$\frac{-16D}{(2n+1)^5 + 4\,(2n+1)}$$

Applying this result to the (complete) sets of (three) terms, leaving out the first (incomplete) set,

we get

$$C = 4D - \frac{4D}{5} - \frac{16D}{3^5 + 4.3} + \frac{16D}{5^5 + 4.5} - \frac{16D}{7^5 + 4.7} + \ldots\ldots\ldots$$

$$C = \frac{16D}{1^5+4.1} - \frac{16D}{3^5 + 4.3} + \frac{16D}{5^5+4.5} - \frac{16D}{7^5+4.7} + \ldots\ldots\ldots\ldots \quad \text{(i)}$$

(ii) Next we consider the identity

$$\frac{1}{2\,(n-1)} - \frac{1}{n} + \frac{1}{2(n+1)} = \frac{1}{n^3 - n}$$

Substituting the odd integers 3, 5, 7, and so on for n and taking the algebraic sum of the corresponding sides, with the positive and negative signs alternating, we have

$$\left(\frac{1}{4} - \frac{1}{3} + \frac{1}{8}\right) - \left(\frac{1}{8} - \frac{1}{5} + \frac{1}{12}\right) + \left(\frac{1}{12} - \frac{1}{7} + \frac{1}{16}\right) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$= \frac{1}{3^3 - 5} - \frac{1}{5^3 - 5} + \frac{1}{7^3 - 7} - \ldots\ldots\ldots\ldots\ldots\ldots$$

The left hand side of the above equation

$$= \frac{1}{4} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots\ldots\ldots\ldots\ldots\ldots$$

We know that

$$C = 4D\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots\ldots\ldots\ldots\ldots\right)$$

$$= 3D + 4D\left(\frac{1}{4} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots\ldots\ldots\right)$$

Which by the result deduced above, gives

$$C = 3D + \frac{4D}{3^3-3} - \frac{4D}{5^3-5} + \frac{4D}{7^3-7} - \ldots\ldots\ldots\ldots \quad \text{(ii)}$$

(ii-a) The following result occurs in the Karana Paddhati (1732 A.D.) In result (ii), starting with the second, group the terms two by two, rewrite the odd numbers 3, 5, 7, 9, 11, ... as (4-1), (4+1), (8-1), (8+1), (12-1), (12+1) and so on, respectively and apply the identity

$$\frac{4}{(4n-1)^3 - (4n+1)} - \frac{4}{(4n+1)^3 - (4n-1)} = \frac{6}{[2.(2n)^2 - 1]^2 - (2n)^2}$$

Then we get

$$C = 3.D + \frac{6.D}{[(2.2^2 - 1)]^2 - 2^2} + \frac{6.D}{[(2.4^2 - 1)]^2 - 4^2} + \frac{6.D}{[(2.6^2 - 1)]^2 - 6^2} \ldots \text{(ii-a)}$$

[32]

(III & IV) grouping the terms of the series

$$C = 4D\left[1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \ldots\ldots\ldots\ldots\right]$$

two by two in 2 different ways, we get

$$(1)\quad C = 4D\left\{\left(1 - \frac{1}{3}\right) + \left(\frac{1}{5} - \frac{1}{7}\right) + \left(\frac{1}{9} - \frac{1}{11}\right) + \ldots\ldots\ldots\right\}$$

$$= 4D\left\{\frac{2}{1.3} + \frac{2}{5.7} + \frac{2}{9.11} + \ldots\ldots\ldots\ldots\right\}$$

$$= 8D\left\{\frac{1}{2^2-1} + \frac{1}{6^2-1} + \frac{1}{10^2-1} + \ldots\ldots\ldots\ldots\right\}\ldots\ldots \text{(iii)}$$

$$(2)\quad C = 4D\left\{1 - \left[\frac{1}{3} - \frac{1}{5}\right] - \left[\frac{1}{7} - \frac{1}{9}\right] - \left[\frac{1}{11} - \frac{1}{13}\right]\ldots\ldots\right\}$$

$$= 4D - \frac{8D}{4^2-1} - \frac{8D}{8^2-1} - \frac{8D}{12^2-1} - \ldots\ldots\ldots\ldots\ldots\ldots \text{(iv)}$$

(v) Consider the identity

$$\frac{1}{2(2n-1)} - \frac{1}{(2n+1)} + \frac{1}{2(2n+1)} = \frac{1}{(2n)^2-1}$$

We substitute the values 1, 2, 3, ... in succession, for n, and take the algebraic sums (separately) of the two sides of the equations so got, with positive and negative signs alternating. The left hand side of the algebraic sum equals

$$\left(\frac{1}{2} - \frac{1}{3} + \frac{1}{6}\right) - \left(\frac{1}{6} - \frac{1}{5} + \frac{1}{10}\right) + \left(\frac{1}{10} - \frac{1}{7} + \frac{1}{14}\right)\ldots$$

$$= \frac{1}{2} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots\ldots$$

and the right hand side of the algebraic sum equals

$$\frac{1}{2^2 - 1} - \frac{1}{4^2 - 1} + \frac{1}{6^2 - 1} - \ldots\ldots\ldots\ldots\ldots$$

Using the above result in

$$C = 4D\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots\ldots\ldots\ldots\ldots\ldots\right)$$

$$2D + 4D\left(\frac{1}{2} - \frac{1}{3} + \frac{1}{5} + \ldots\ldots\ldots\ldots\ldots\ldots\right)$$

We get

$$C = 2D + \frac{4D}{2^2 - 1} - \frac{4D}{4^2 - 1} + \frac{4D}{6^2 - 1} - \ldots\ldots\ldots\ldots\ldots\ldots \quad \text{(v)}$$

For the remainder term of the series (v) see below)

Mādhavan's formulae for the remainder terms

The sixth chapter of the *Yukti Bhasa* concludes with the following results of Madhavan,

$$C \sim 4D - \frac{4D}{3} + \frac{4D}{5} - \ldots\ldots\ldots\ldots\ldots \pm \frac{4D}{2n-1} \mp 4D.\frac{n}{(2n)^2 + 1} \quad \text{(G)}$$

$$C \sim 4D - \frac{4D}{3} + \frac{4D}{5} - \ldots\ldots\ldots\ldots \pm \frac{4D}{2n-1} + 4D.\frac{n^2+1}{[(n^2+1)4+1)]n} \quad \ldots\ldots \quad (H)$$

The *Yukti-bhasa* exhibits a continued fraction, namely

$$\frac{1}{4n+}\ \frac{4}{.\ 4n+}\ \frac{16}{4n}$$

the second and third convergents of which reduced to lowest terms give the last (or remainder) terms in (G) and (H).

The method used to get the remainder terms is one of trial and error. An outline of the procedure is given below:-

Consider the result:

$$C = 4D\ (1 - \frac{1}{3} + \frac{1}{5} - \ldots\ldots\ldots\ldots \pm \frac{1}{2n-1} \pm A_n)$$

Where A_n is the remainder after n terms of the series.

Replace n by (n - 1) and subtract the expression obtained from the one above. If we can choose an A_n such that

$$A_n + A_{n-1} - \frac{1}{2n-1} = 0, \text{ that would be ideal.}$$

Setting $A_n = \frac{1}{4n-1}$ gives

$$A_n + A_{n-1} - \frac{1}{2n-1} = \frac{-4n+5}{(2n-1)\ (4n-3)\ (4n+1)}$$

While $A_n = \frac{1}{4n}$ gives

$$A_n + A_{n-1} - \frac{1}{2n-1} = \frac{1}{(4n\,(n-1)\,(2n-1)}$$

So that for large n, $A_n = \frac{1}{4n}$ gives a much smaller error.

Now compare in the same way the error terms

$$A_n = \frac{1}{4n +} \quad \frac{1}{4n}$$

giving

$$A_n + A_{n-1} - \frac{1}{2n-1} = \frac{48\,n^2 - 48n - 13}{(2n-1)\,[\,16\,(2n-1)^4 - 24(2n-1)^2 + 9]}$$

While $A_n = \frac{1}{4n+} \quad \frac{4}{4n}$

giving

$$A_n + A_{n-1} - \frac{1}{2n-1} = \frac{-4}{[(2n-1)^5 + 4\,(2n-1)]}$$

So the second approximation is better, with the error for

$A_n + A_{n-1} - \frac{1}{2n-1}$ of order n^{-5} as opposed to n^{-3}. In fact

any number here other than 4 will give an approximation for which the error is of the order of n^{-3}, so that 4 is in fact the best. Note that the latter value of A_n reduced to lowest terms, is the remainder term in (G).

An entirely identical analysis shows that

$$A_n = \cfrac{1}{4n + \cfrac{4}{4n + \cfrac{16}{4n}}}$$

gives $A_n + A_{n-1} - \frac{1}{2n-1}$

$$= \frac{36}{(2n-1)^7 + 7 \cdot (2n-1)^5 + 28 \cdot (2n-1)^3 - 36 \cdot (2n-1)}$$

Hence the error is of the order n^{-7}, with any number other than 16 giving a larger error term. This value of A_n reduces to the remainder term in (H).

The *Yukti Bhasa* states that a circle of diameter *"Anunanutnananananunnanityam"*, which (in the katapayadi system) denotes the number 10, 000, 000, 000 has its circumference given by *"Candamsu candradhamakambhipalah"* i.e. 31, 415, 926, 536.

We get this result from (G) for the value $n = 28$.

The Yukti - Bhasa contains another series with the remainder term. (See Part I - Jyestha Devan - v), It is given by:

$$\pi \sim 2 + \frac{4}{2^2-1} - \frac{4}{4^2-1} + \ldots\ldots\ldots\ldots\ldots + \frac{4}{(2n)^2-1} - \frac{4}{2[(2n+1)^2+2]}$$

Analysis similar to the one discussed above, shows that

$$A_n = \frac{4}{[(2n+1)^2 + x]} \text{ gives}$$

$$A_n + A_{n-1} - \frac{1}{(2n)^2-1}$$ is of order n^{-6} when x = 2 and of order n^{-4} for any other value of x. Hence x = 2, gives the best approximation

Madhavan's infinite series for the sine and cosine

Chapter seven of the Yukti bhasa deals mainly with the study of the sine and cosine of an arc of a circle.

Starting with the results:

(i) $R \sin\frac{\pi}{6} = \frac{1}{2} R$ and (ii) $R \sin\frac{\pi}{2} = R$

and applying the formulae:

$$R.\sin A = \frac{1}{2}\sqrt{(R.\sin 2A)^2 + R^2.(1 - \cos 2A)^2}$$ and

$$R.\sin\left(\frac{\pi}{2} - A\right) = \sqrt{R^2 - (R.\sin A)^2}$$

the values of R.sin rx, where $x = \frac{\pi}{48}$ and r=1, 2, ... 24 are found.

It was known that the cosine of an arc of a circle is equal to the sine of the complementary arc (i.e., the portion of the arc to be traversed to complete a quadrant). Thus the values of R sin rx and R cos rx where $x = (\pi / 48)$ and $r = 1, 2, \ldots\ldots\ldots\ldots 24$, are derived.

Next procedure is the calculation of the sine and cosine of any arc of a circle i.e., an arc whose length is not an exact multiple of $(\pi R/48)$. The arc-length can be anything between 0 and $2\pi R$. Let us assume that all arcs begin at the initial point; then the terminal point can be in any one of the four quadrants. In order to get the sine and cosine of an arc of any length, it is enough if we find the sine and cosine of the portion of the arc that lies in the quadrant where the terminal point lies. This follows from the statement that the sine and cosine of an arc whose terminal point lies in an even (i.e., the second or fourth) quadrant are numerically equal to the cosine and sine of the portion of the arc lying in that (even) quadrant respectively; also the sine and cosine of an arc whose terminal point lies in an odd (the first or the third) quadrant are numerically equal to the sine and cosine of the arc lying in that (odd) quadrant respectively[(2)]. Hence we need confine our attention to arcs of a circle lying in the first quadrant.

Our objective, here, is to find the sine and cosine of the arc AE (Figure - vi). Choose a point B on the circle, such that the length of the arc AB is an exact multiple of $(\pi R / 48)$ and the length of the arc BE does not exceed $(R \pi/96)$. There is no loss of generality in assuming this, because if the length of the arc BE exceeds $(\pi R / 96)$ then the length of the arc to be traversed from E along the arc EP to reach the point whose arc-length is $(\pi R / 48)$ from B, will be less than $(R / 96)$; here the small arc will have to be subtracted). Then the line-segments OD = BH and OH = DB are the sine and cosine of arc AB respectively.

(2) Ignoring the signs, this boils down to:
$\sin(90 + x) = \sin(270 + x) = \pm \cos x$ and
$\sin(180 + x) = -\sin x$ and similarly for cosines.

Let K be the midpoint of the arc EB; then MK (= OL) is the sine and LK (=OM), the cosine of the arc AK. The triangles NEB and LKO are similar since the two triangles have their corresponding sides (in the order mentioned) mutually perpendicular. Hence

$$\frac{NE}{LK} = \frac{EB}{KO} = \frac{BN}{OL}$$

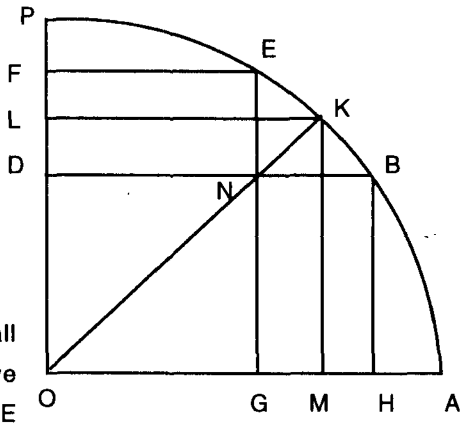


Fig. VI

$$\text{So} \quad NE = \frac{LK \cdot EB}{KO}$$

$$\text{and } BN = \frac{OL \cdot EB}{KO}$$

The arc BE, being very small (not exceeding π R / 96) we can assume that the arc BE and its full-chord are equal; further the points K and B on the circle are close to each other and so we can take BD and OD equal to LK and OL respectively (as first approximations.

$$\text{Hence } NE = \frac{BD. \text{ arc } EB}{R} \qquad \text{and } BN = \frac{OD \text{ . arc } EB}{R}$$

Now, sine of arc AE = GE = GN + NE = HB + NE

$$= \text{sine of arc AB} + \frac{\text{cosine of arc AB . arc BE}}{R}$$

and cosine of arc AE = FE = OG = OH - GH = DB - NB

$$= \text{Cosine of arc AB} - \frac{\text{sine of arc AB . arc BE}}{R}$$

Assume the length of arc BE to be h; then the length of the arc BK = (h / 2). Further denote the sine and cosine of arc AB respectively by b and k, then the first approximation, given above, can be rewritten as:

$$\text{sine of arc AE} = b + \frac{k.\ h}{R} \quad \text{and cosine of arc AE} = k - \frac{b.\ h}{R}$$

Note that, in order to get the sine and cosine of arc AE, from those of AB we have to add or subtract a term that depends on the arc BE; hence the smaller the arc, the closer the result will be. Let us proceed with an arc of length h / 2 instead of h.

$$\text{Now the sine of arc AK} = b + \frac{k\ .\ (h\ /\ 2)}{R}$$

$$= b + \frac{k}{D}\text{, where } D = \frac{2R}{h}$$

$$= b_1\text{, say.}$$

$$\text{and the cosine of arc AK} = k - \frac{k\ .\ (h\ /\ 2)}{R}$$

$$= k - \frac{b}{D} = k_1\text{, say}$$

To get the sine and cosine of arc AE, we had to use the values of NE and NB; therein we took k and b as approximations to LK and OL, the cosine and sine of arc AK.

Now we can use the values k_1 and b_1 and get

$$NE = \frac{k1 \,.\, h}{R} = \frac{2\,k_1}{D} \;\&\; NB = \frac{b_1 \,.\, h}{R} = \frac{2b_1}{D}$$

Now

the sine of arc AE = GE = GN + NE = $b + \dfrac{2k_1}{D}$

$$= b + \left(k - \frac{b}{d}\right) \,.\, \frac{2}{D}$$

and the cosine of arc AE = FE = DB - NB = $k - \dfrac{2b_1}{D}$

$$= k - \left(b + \frac{k}{D}\right) \,.\, \frac{2}{D}$$

Taking the length of the arc AB to be R.x, we get

$$R.\sin(x + h) = R.\sin x + \left(R.\cos x - \frac{R.\sin x}{D}\right) \,.\, \frac{2}{D}$$

and

$$R.\cos(x + h) = R.\cos x - \left(R.\sin x + \frac{R.\cos x}{D}\right) \,.\, \frac{2}{D}$$

These two results were enunciated by Mādhavan (c. 1340 - c. 1425).

If the arc for which we want to find the sine and cosine falls short of an exact multiple of (π / R / 48) by less than (π R / 96), proceed in the same manner but remember that the corrections have to be subtracted

from the sine and added to the cosine of the arc, whose length is an exact multiple of (π R / 48).

The following conventions and notation, with respect to Figure-vii, will be convenient for the discussion that ensue:

(a) PQ is an arc of a circle of radius R

(2) the arc PQ is divided into n equal parts by points

$P_1, P_2, P_3, \ldots\ldots P_{n-1}, P_n$ (=Q)

(3) $Q_1, Q_2, Q_3 \ldots\ldots Q_n$ are the midpoints of the arcs

$PP_1, P_1P_2, P_2P_3, \ldots\ldots P_{n-1}Q$ respectively

(4) the perpendiculars from P_r to OX and OY meet them

at L_r and M_r respectively, for r = 1, 2, 3, ...n.

(5) the perpendiculars from Q_r to OX and OY meet them at

J_r and N_r respectively, for r = 1 , 2, 3, ...n.

(6) the arcs $PP_1, P_1P_2, \ldots\ldots, P_{n-1}Q$ are denoted by

$S_1, S_2, \ldots\ldots, S_n$ respectively and each of those small arcs = s.

(7) the sine and cosine of the arc S_r are denoted by b_r and k_r respectively for r = 1, 2, 3, ...n.

(8) c denotes the length of the full chord of an arc of length s;

S_r the arc length PP_r and B_r and K_r its respective sine and cosine

(9) (i) $S_r = s_1 + s_2 + s_3 + \ldots\ldots\ldots\ldots\ldots\ldots + s_r$

(ii) $B_r = b_1 + b_2 + b_3 + \ldots\ldots\ldots\ldots\ldots + b_r$ for r = 1, 2, 3,..., n.

further $S_n = S$, $B_n = B$ and $K_n = K$.

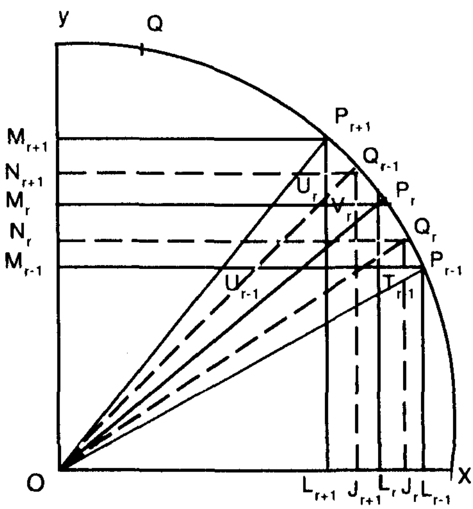
(10) The versine of the arc S_r = R - cosine of the arc S_r

Now each of the three sets of triangles

$(P_{r+1} U_r P_r)$, $(O T_{r+1} Q_{r+1})$, $(P_r J_{r-1} P_{r-1}$, $)O J_r Q_r)$,

and $(V_r Q_r Q_{r+1}$, $L_r P_r O)$ forms a similar pair, since the corresponding sides (in the order given) in each pair mutually perpendicular. Hence

Figure - vii



$$\frac{P_{r+1} U_r}{O J_{r+1}} = \frac{U_r P_r}{J_{r+1} Q_{r+1}} = \frac{P_r P_{r+1}}{Q_{r+1} O} ;$$

So $P_{r+1} U_r = O J_{r+1} . \frac{C}{R}$

$$\frac{P_r T_{r-1}}{O J_r} = \frac{T_{r-1} P_{r-1}}{J_r Q_r} = \frac{P_{r-1} P_r}{Q_r O} ;$$

So $P_r T_{r-1} = O J_r . \frac{C}{R}$ and

$$\frac{V_r Q_r}{L_r P_r} = \frac{Q_r Q_{r+1}}{P_r O} = \frac{Q_{r+1} V_r}{O L_r}$$

So $V_r Q_r = L_r P_r . \frac{C}{R}$ (J)

Since each of the chords $P_r P_{r+1}$, $P_{r-1} P_r$ & $Q_r Q_{r+1}$ is the full chord of an arc of length s is equal to c. Further

$P_{r+1} U_r = b_{r+1}$, $P_r T_{r-1} = b_r$, & $L_r P_r = B_r$.

Now,

$b_r - b_{r+1} = P_r T_{r-1} - P_{r+1} U_r = (O J_r - O J_{r+1}) . \frac{C}{R}$

$$V_r Q_r \cdot \frac{c}{R} = (L_r P_r \cdot \frac{c}{R}) \cdot \frac{c}{R} \quad . \quad = B_r \cdot (\frac{c}{R})^2$$

Thus

$$\frac{b_r - b_{r+1}}{B_r} \quad \left[\frac{C}{R}\right]^2 \text{ for } r = 1, 2, \ldots\ldots\ldots\ldots, (n-1) \quad (K)$$

(3)

Similary, we can get

$$\frac{k_{r+1} - k_r}{K_r} = \left[\frac{C}{R}\right]^2 \text{ for } r = 1, 2, \ldots\ldots\ldots\ldots (n-1).$$

In the *Yukti-bhāsa* this result has been demonstrated for the various cases of sine arising out of n = 3.

We devote the rest of the discussion to get approximations to and refinements on the values of the sine and cosine in terms of powers of the arc-length.

---

(3) This result is now, a routine problem in trigonometry:

(R.sin 2rx - R. sin (2r -2)x) - (R. sin (2r + 2)x - R. sin 2rx)

= 2R.sin 2rx - R sin (2r + 2) x + sin (2r - 2) x

= R.sin 2rx 2 (1 - cos 2x)

$$= R.\ \sin 2rx. \left[\frac{2R\ .\ \sin x}{R}\right]^2$$

similary for the cosine function.

Section i

Here we present a method to get an approximation for B in terms of S. Now, $b_1 = B_1 - 0$, $b_2 = B_2 - B_1$, ... $b_r = B_r - B_{r-1}$ ....
Hence,

$b_1 - b_2 = B_1 - (B_2 - B_1)$, $b_2 - b_3 = (B_2 - B_1) - (B_3 - B_2)$ ....

$$b_r - b_{r+1} = (B_r - B_{r-1}) - (B_{r+1} - B_r) \quad ....$$

Let us denote the differences between the b's by d's:

$$d_1 = b_1 - b_2 = B_1 - (B_2 - B_1)$$

$$d_2 = b_2 - b_3 = (B_2 - B_1) - (B_3 - B_2)$$

$$d_3 = b_3 - b_4 = (B_3 - B_2) - (B_4 - B_3)$$

— —

— —

— —

$$d_{n-1} = b_{n-1} - b_n = (B_{n-1} - B_{n-2}) - (B_n - B_{n-1})$$

Therefore, $d_1 + d_2 + ... + d_{n-1} = b_1 - b_n = B_1 - (B_n - B_{n-1})$

or

$$b_n = B_1 - (d_1 + d_2 + d_3 + ... + d_{n-1})$$

We have by (K)

$$d_r = b_r - b_{r+1} = B_r \cdot \left[\frac{C}{R}\right]^2 \qquad \text{.........} \text{(L)}$$

Now $b_1 = B_1$

$$b_2 = B_2 - B_1 = B_1 - [(B_1 - 0) - (B_2 - B_1)] = B_1 - B_1 \cdot \left[\frac{C}{R}\right]^2$$

$$b_3 = B_3 - B_2 = (B_2 - B_1) - [(B_2 - B_1) - (B_3 - B_2)] =$$

$$B_1 - B_1 \cdot \left[\frac{C}{R}\right]^2 - B_2 \left[\frac{C}{R}\right]^2 \text{.................................}$$

....

....

$$b_n = B_n - B_{n-1} = B_1 - B_1 \cdot \left[\frac{C}{R}\right]^2 - B_2 \left[\frac{C}{R}\right]^2 - B_{n-1} \left[\frac{C}{R}\right]^2$$

So $B_n = b_1 + b_2 + \text{....................} + b_n$

$$= n \cdot B_1 - [(n-1) \cdot B_1 + (n-2) \cdot B_2 + \text{........} + 2 \cdot B_{n-2} + B_{n-1}] \cdot \left[\frac{C}{R}\right]^2$$

OR

$$[(n-1) \cdot B_1 + (n-2) \cdot B_2 + \text{............} + 2 \cdot B_{n-2} + B_{n-1}] \cdot \left[\frac{C}{R}\right]^2 \left(n.B_1 - B_n\right) =$$

Also, by (L) $\quad d_r = B_r \,.\, \left[\frac{C}{R} \right]^2$

Hence $(n-1) \,.\, d_1 + (n-2) \,.\, d_2 + \ldots\ldots + 2.d_{n-2} + d_{n-1} =$

$$[(n-1) \,.\, B_1 + (n-2) \,.\, B_2 + \ldots\ldots\ldots\ldots + 2 \,.\, B_{n-2} + B_{n-1}] \,.\, \left[\frac{C}{R} \right]^2 =$$

$$n \,.\, B_1 - B_n \neq n \,.\, B_1 - B$$

When n is very large the segments of arc become very small and so we can take $B_\gamma = r \,.\, c$, $c = s$ and n.s. = S, as first approximation. Then, the above equation reduces to:

$$B \sim S \;\; \frac{S^3}{3! \,.\, R^2} \quad \text{, since } (n-1) \,.\, 1 + (n-2) \,.\, 2 + \ldots\ldots\ldots$$

$$+ 1 \,.\, (n-1) \sim \frac{n^3}{3!}$$

Section ii

Next we present a sequence of refinements to the approximation for the sine and versine of an arc (i.e., R - cos of an arc). By Fig. - vii and (J)

$$J_{r+1} \; J_r = V_r \, Q_r = L_r \, P_r \,.\, \frac{C}{R} = B_r \,.\, \frac{C}{R}$$

Hence

$$\sum_{r=1}^{m} J_{r+1} \quad J_r \sum_{r=1}^{m} B_r \quad \frac{c}{R} \qquad \text{or}$$

$$J_1 J_{m+1} = \sum_{r=1}^{m} B_r \cdot \frac{C}{R}$$

So versine of arc $S_m = \sum_{r=1}^{m} B_r \frac{C}{R}$ .... (M)

Now by Figure - vii, we have $P_r T_{r-1} = O J_r . \frac{C}{R}$

$b_1 = O J_1 . \frac{C}{R}$ and $b_r = O J_r . \frac{C}{R}$

So

$b_1 - b_r = (OJ_1 - OJ)_r . \frac{C}{R} = J_r J_1 . \frac{C}{R}$

$= L_r P . \frac{C}{R}$, since the segments of arc are very small

$=$ versome pf arc $S_r . \frac{C}{R}$

Hence

$(b_1 - b_n) + (b_1 - b_{n-1}) + (b_1 - b_{n-2}) +$ .... $= n b_1 - B_n$

$= [\sum_{r=1}^{n}$ versine of arc $S_r] . \frac{C}{R}$

But by (M)

Versine of arc $S_r = (B_1 + B_2 +$ .... $+ B_r) . \frac{C}{R}$

B_1 , B_2 , B_3 ..... which occur in the above equation are not known;

but we know that n is very large and so can take B_r = r.s as a first approximation for r = 1, 2, 3, ... , n;

further the arcs s_1, s_2, s_3............. s_n are very small and so we can take C = S as well.

So versine of arc S_r = $(s + 2.\, s + 3\,.\, s + \ldots\ldots + r\,.\, s)\,.\,\frac{s}{R}$

$$= \frac{r\,(r+1)\,.\,s}{2}\cdot\frac{s}{R}$$

So $n.\,b_1 - (b_n + b_{n-1} + \ldots\ldots\ldots\ldots) = \left[\sum_{r=1}^{n}\frac{r(r+1)}{2}\cdot\frac{s^2}{R}\right]\cdot\frac{s}{R}$

Using the result established in the *Yukti Bhāsa*(4) viz.

---

(4) In the *Yukti-bhasa* we come across the following working:

$$\sum_{r=1}^{n} r = \frac{n\,.\,(n+1)}{2} \sim \frac{n^2}{2} \quad \text{when n is very large.}$$

$$\sum_{r=1}^{n} \frac{r\,.\,(r+1)}{2} = \sum_{r=1}^{n}\left[\frac{r^2}{2}+\frac{r}{2}\right] = \frac{n(n+1)(2n+1)}{2 \times 6} + \frac{n(n+1)}{2 \times 2}$$

$$\frac{n\,(n+1)\,(n+2)}{3!} \sim \frac{n^3}{3!} \quad \text{for very large n.}$$

$$\sum_{r=1}^{n} \frac{r\,.\,(r+1)\,.\,(r+2)}{3i} \quad \sum_{r=1}^{n} \frac{r^3}{6} + \frac{r^2}{2} + \frac{r}{3}$$

$$\frac{n^2\,(n+1)^2}{6 \times 4} + \frac{n\,(n+1)\,(2n+1)}{2 \times 6} + \frac{n\,(n+1)}{3 \times 2}$$

$$\frac{n\,(n+1)\,(n+2)\,(n+3)}{4!} \sim \frac{n}{4!} \quad \text{for very large n.}$$

$$\sum_{r=1}^{n} \frac{r\,(r+1)}{2} \sim \frac{n^3}{3!}$$

We get

$$n\,.\,b_1 - B_n \sim \frac{(n\,.\,s)^3}{3\,!\,.\,R^2}$$

$$\text{or} \quad B_n = n\,.\,s - \frac{(n\,.\,s)^3}{3\,!\,.\,R^2} = S - \frac{S^3}{3\,!\,.\,R^2} \quad \text{approximately}$$

In calculating the versine of arc S with the formula

$$\text{versine of arc } S_n = \frac{s}{R}\,(s + 2\,.\,s + 3\,.\,s + \ldots\ldots\ldots\ldots + n\,.\,s)$$

s was used in place of the b's. Hence the result is a rough approximation. If the corrected value, obtained above be used, then

$$\text{versine of arc } S_n = \frac{s}{R}\left\{\left(ns - \frac{n^3\,.\,s^3}{3\,!\,.\,R^2}\right] + \left[\,(n-1)s - \frac{(n-1)^3\,.\,s^3}{3\,!\,.\,R^2}\right] + \ldots\right\}$$

Hence the correction to be made on the versine

$$= \frac{s}{R}\,.\,\frac{s^3}{3\,!\,.\,R^2}\,[n^3 + (n-1)^3 + \ldots\ldots\ldots\ldots\ldots\ldots]$$

$$\sim \frac{s^4}{3\,!\,.\,R^3}\,.\,\frac{n^4}{4} = \frac{n^4\,s^4}{4\,!\,.\,R^3}$$

Note that versine of arc $S_n = \left[\sum_{r=1}^{n} B_r \right] \cdot \frac{s}{R}$

and

$$s - B = n . s - B \sim n . b_1 - B_n = \left[\sum_{r=1}^{n} \text{versine of arc } S_r \right] \cdot \frac{s}{R}$$

These two equations give the successive refinements to be made on S-B and versine respectively.

Thus the required sine = B = n . s = S (first approx .)

$$\text{versine} = \frac{s}{R} \, [\, n . s + (n+1) . s + ...] = \frac{n^2 . s^2}{2 . R} = \frac{s^2}{2 . R} \text{ (Ist approx.)}$$

$$\text{Ist correction to the sine} = \frac{(n\,s)^2 + (n-1)^2 . s^2 + ...}{2! . R} \cdot \frac{s}{R}$$

$$\sim \frac{n^3 . s^3}{3 ! . R^3} = \frac{S^3}{3 ! . R^3}$$

$$\text{Ist correction to versine} = \frac{(\,ns\,)^3 + (\,n - 1\,)^3 . s^3 + ...}{3 ! . R^2} \cdot \frac{s}{R}$$

$$\sim \frac{n^4 . s^4}{4 ! . R^3} = \frac{S^4}{4 ! . R^3}$$

$$\text{2nd correction to the sine} = \frac{(\,ns\,)^4 + (\,n - 1\,)^4 . s^4 + ...}{4 ! . R^3} \quad \frac{s}{R}$$

$$\sim \frac{n^5 . s^5}{5\,!\,.R^4} \cdot \frac{S^5}{5\,!\,.\,R^4}$$

$$\text{2nd correction to versine} = \frac{(ns)^5 + (n-1)^5 . s^5 + \ldots\ldots}{5\,!\,.\,R^4} \qquad \frac{s}{R}$$

$$\sim \frac{n^6 . s^6}{6\,!\,.R^5} \cdot \frac{S^6}{6\,!\,.\,R^5}$$

and so on.

In all these a was taken in place of the b's; hence the absolute values of the correction terms are greater than the correct values. So every correction term should be subtracted from the preceding term, this result should be subtracted from the correction term before that and so on. This process can be continued on and on.

$$\text{Now sine of arc S} = B = S - \left[\frac{S^3}{3\,!\,.\,R^2} - \left[\frac{S^5}{3\,!\,.\,R^4} - \left[\frac{S^7}{3\,!\,.\,R^6} \ldots\right]\right]\right]$$

$$= S - \frac{S^3}{3\,!\,.\,R^2} + \frac{S^5}{5\,!\,.\,R^2} - \frac{S^7}{7\,!\,.\,R^6} + \ldots\ldots$$

$$\text{versine of arc S} = \frac{S^2}{2!\,.\,R} - \left[\frac{S^4}{4\,!\,.\,R^3} - \left[\frac{S^6}{6\,!\,.\,R^5} - \ldots\ldots\right]\right]$$

$$= \frac{S^2}{2!\,.\,R} - \frac{S^4}{4!\,.\,R^3} + \frac{S^6}{6!\,.\,R^5} - \cdots\cdots$$

cosine of arc S = K = R - versine of arc S.

Series expansion for the square of the sine

The Yukti-bhāsa quotes a verse giving the series expansion for the square of the sine of an arc of a circle. If R denotes the radius of the circle, S, the given arc and B its sine, then:-

$$B^2 = S^2 - \frac{S^4}{\left[2^2 - \frac{2}{2}\right] R^2} + \frac{S^6}{\left[2^2 - \frac{2}{2}\right]\left[3^2 - \frac{3}{2}\right] R^4} - \frac{S^8}{\left[2^2 - \frac{2}{2}\right]\left[3^2 - \frac{3}{2}\right]\left[4^2 - \frac{4}{2}\right] R^6} + \ldots$$

This result is obtained by just expanding the square of the infinite series for B, the sine of arc S, and regrouping the terms.

Denoting B by R.sin x and S by R.x, the above result reduces to:-

$$\sin^2 x = x^2 - \frac{x^4}{\left[2^2 - \frac{2}{2}\right]} + \frac{x^6}{\left[2^2 - \frac{2}{2}\right]\left[3^2 - \frac{3}{2}\right]} - \frac{x^8}{\left[2^2 - \frac{2}{2}\right]\left[3^2 - \frac{3}{2}\right]\left[4^2 - \frac{4}{2}\right]} + \ldots\ldots\ldots\ldots$$

The pattern exhibited by the successive denominators is lovely.

(Note that the coefficient of x^{2n} in the above series is

$$(-1)^{n-1} \cdot \frac{1}{\left[2^2 - \frac{2}{2}\right]\left[3^2 - \frac{3}{2}\right]\left[4^2 - \frac{4}{2}\right] \cdots\cdots\cdots\cdots \left[n^2 - \frac{n}{2}\right]}$$

which can be simplified as

$$(-1)^{n-1} \cdot \frac{1}{\left[2 - \frac{1}{2}\right] . 2\left[3 - \frac{1}{2}\right] . 3\left[4 - \frac{1}{2}\right] .4 \ldots \left[n - \frac{1}{2}\right] .n}$$

Multiplying each of the (2n-2) factors in the denominator by 2 and the numerator by $2^{(2n-2)}$ we get

$$(-1)^{n-1} \cdot \frac{2^{(2n-2)}}{3 . 4 . 5 . 6 . 7 . 8 \ldots\ldots\ldots\ldots\ldots\ldots (2n-1) . 2n} =$$

$$= \frac{1}{2} . (-1)^{n+1} \frac{2^{2n}}{(2n)!}$$

the coefficient of x^{2n} in the expansion of (1 / 2) . (1 - cos 2x).

PARAMESVARAN and Cyclic Quadrilateral

In the historical section, we gave a formula, enunciated by Paramesvaran (c. 1360 - c. 1460), for the radius R, of the circle circumscribing a cyclic quadrilateral, in terms of its sides a, b, c, d.

The *Yukti Bhāsa* gives a demonstration of this formula, starting from fundamentals. Let us follow the procedure in the Yukti Bhāsa.

1. *Some properties of triangles :*

Proposition I : (Pythagoras theorem) Let ABC be a right triangle, right angled at A. Then

$$CA^2 + AB^2 = CB^2$$

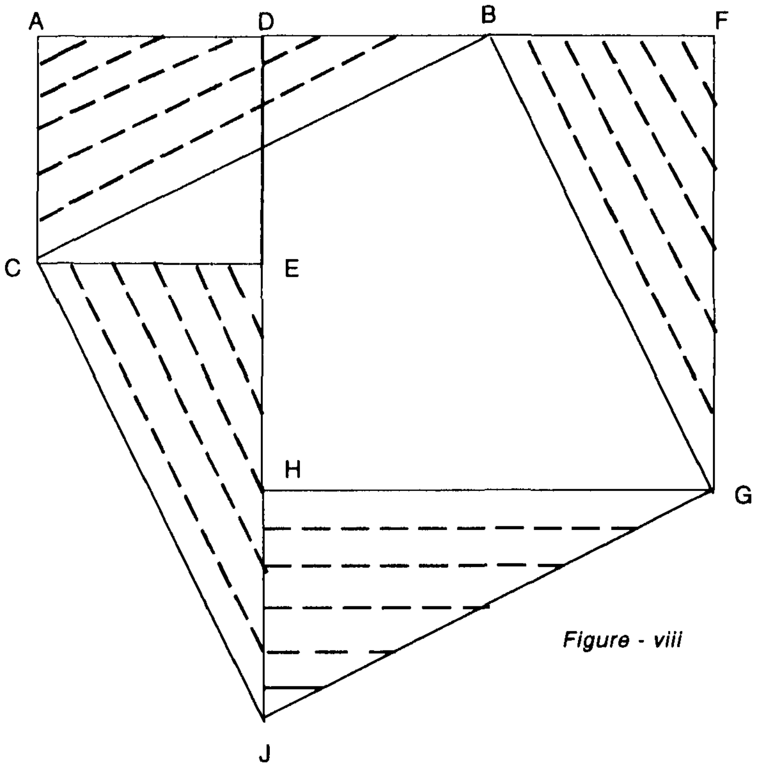


Figure - viii

Construction : Draw a square ACED on AC and a square DHGF on DF, where DF = AB. (Fig. viii)

Proof : Since DF = AB, we have BF = DF - DB = AB - DB = AD. We know that FG = AB. Hence the hypotenuse BG = hypotenuse CB.

Now rotate the triangle FGB about the vertex G, clockwise (through 270 degrees), then the triangle takes the position of HGJ.

So HJ = BF and hyp. GJ = hyp . BG = hyp. CB.

Now JE = JH + HE = ED + HE = HD = AB, and CE = CA.

Now we rotate the triangle ACB about the vertex C, anticlockwise (through 270 degrees), then the triangle takes the position of ECJ.

The squares ACED and DHGF together transformed into the square CJGB.

Hence square ACED + square DHGF = square CJGB or

$$CA^2 + AB^2 = CB^2$$

Proposition II : -

Area of a triangle = (1/2) the base x altitude

Proof : Let ABC be a triangle and AD its altitude.

We have to prove that :

area of the triangle ABC = (1 / 2) BC x AD. (Fig. - ix) Let E, F be the midpoints of the sides BA and AC respectively. Draw perpendiculars from E and F to BC to meet it

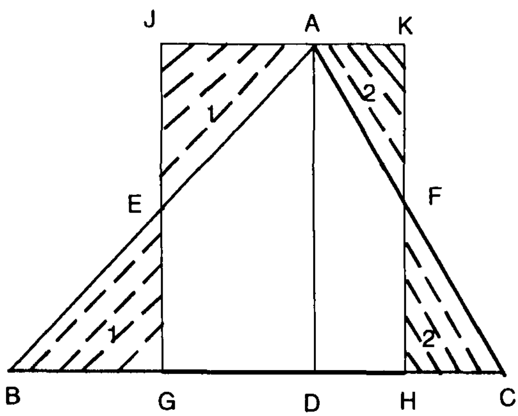


Figure - ix

at G and H respectively.

Now BG = GD (1 / 2) BD ; and DH = HC = (1 / 2) DC.

Hence GD + DH = GH = (1 / 2) BC.

The triangle EGB is lifted and placed so that B falls on A, BE along AE and G on J. Similarly the triangle FHC is lifted and placed that C falls on A, CF along AF and H on K. Now the triangle ABC has transformed into the rectangle GHKJ.

So, area of the triangle ABC = area of the rectangle GHKJ.

= GH x GJ = (1 / 2) BC x AD

Proposition III :

In a triangle , the product of any two sides divided by the diameter of the circle circumscribing the triangle = the altitude on the third side.

Let ABC be a triangle with base AC and O, the center of the circle circumscribing the triangle. (Fig. x)

Join the midpoint of BC to O and produce it both ways to meet the circle at the points E and F. Then EF will be a diameter of the circle. Starting with the point E draw a parallel to AC, to meet the circle at G. Mark D, the mid-point of the arc AC.

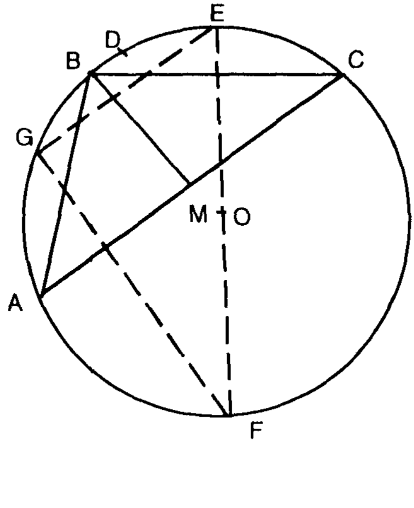


Figure - x

Proof:

Now arc AD = arc DC and arc AG = arc EC.

So arc GD = arc AD - arc AG = arc AD - arc EC

= arc AD - (1 / 2) arc BC

and arc DE = arc DC - arc EC = arc DC - (1 /2) arc BC.

Hence, arc GD + arc DE = arc AD + arc DC - arc BC = arc AB

or arc GE = arc AB. Hence GE = AB.

The two right triangles EGF and BMC have their respective sides perpendicular ; so they are similar.

Hence $\frac{FE}{EG} = \frac{BC}{BM}$; or $BM = \frac{EG \times BC}{FE} = \frac{AB \times BC}{2R}$

Where R is the radius of the circle circumscribing the triangle ABC.

2. *Some Properties of the chords of a circle :*

Lemma I : The difference of the squares on the chords of any two arcs of a circle = the product of the chord of an arc equal to the sum of the two arcs and the chord of an arc equal to the difference of the two arcs.

Proof: Let AB and BC be any two arcs of a circle. Figure-xi (Note that there is no loss of generality in assuming the arcs have a common point B). Let AB be the greater arc. On the arc AB mark a point D so that arc AD = arc BC. Now arc DB = arc AB - arc AD = arc AB - arc BC. i.e., the difference of the two arcs and arc AC = arc AB + arc BC = sum of the two arcs. We have to prove that

$$AB^2 - BC^2 = AC \times DB.$$

Arc AD = arc BC.

So AD = BC.

From the points B and D draw perpendiculars to AC to meet it at E and F respectively.

ACBD is a cyclic quadrilateral and the sides AD and BC are equal ; so DF = BE. Further DB = FE.

In the triangle ABC,

$$AB^2 - BC^2 = (AE^2 + EB^2) - (BE^2 + EC^2) = AE^2 - EC^2$$

$$= (AE + EC) . (AE - EC) = (AE + EC) . (AE - AF)$$

$$= AC . FE = AC . DB.$$

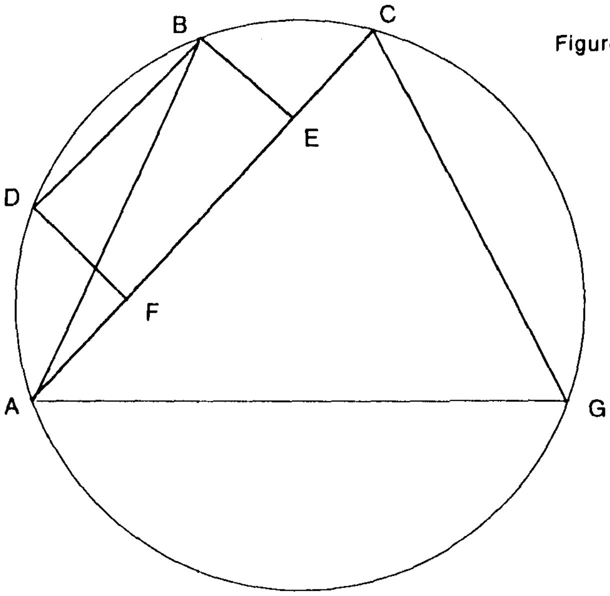


Figure - xi

Lemma II : The product of any two chords of a circle is equal to the difference between the squares on the chord of the arithmetic mean of the arcs and the chord of the semi-difference of the arcs [5].

---

(5) The results contained in the lemmas can be put in the modern trigonometic language thus:

$\sin^2 A - \sin^2 B = \sin (A+B) . \sin (A-B)$ and

$\sin A . \sin B = \sin^2 \frac{A+B}{2} - \sin^2 \frac{A-B}{2}$)

Derivation of lemma II from lemma I :

Consider the two chords AC and DB (Figure xi).

Now arc AC + arc DB = (arc AB + arc BC) + (arc AB - arc AD)

= 2 arc AB , (since arc BC = arc AD)

and arc AC - arc DB = (arc AD + arc DB + arc BC) - (arc DB)

= 2 arc BC, (since arc AD = arc BC)

$$\text{So arc AB} = \frac{\text{arc AC + arc DB}}{2} \quad ; \text{arc BC} = \frac{\text{arc AC + arc DB}}{2}$$

Now AC . DB = (AE + EC) (AE - AF) = (AE + EC) (AE - EC)

$= (AE^2 - EC^2) = AB^2 - BC^2$

= square on the chord of (1 / 2) (arc AC + arc DB)

= square on the chord of (1 / 2) (arc AC - arc DB)

3. *Some Properties of a cyclic quadrilateral :*

(a) Formulae for the diagonals of a cyclic quadrilateral :

Let ABCD be a cyclic quadrilateral with its sides AB > BC > AD > CD.

In Figure - xii, we take the points E and F on the arc

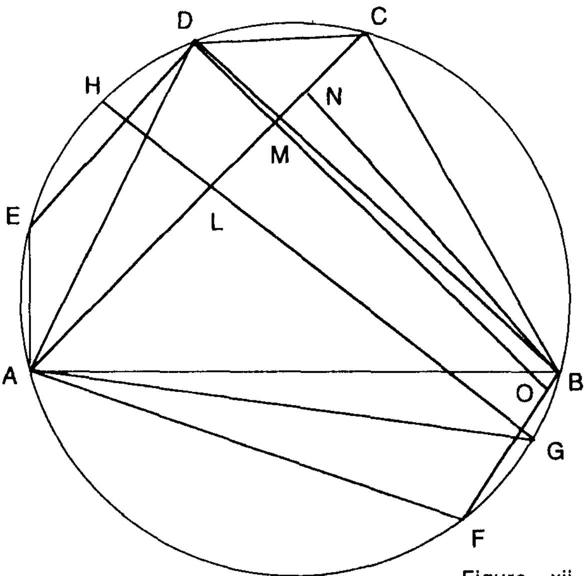


Figure - xii

AD and arc AB resp., so that arc AE = arc CD and arc AF = arc CB.

H and G, the midpoints of arcs ED and FB resp.,

Now arc HE + arc EA + arc AF + arc FG = arc HD + arc DC + arc CB + arc BG

i.e., arc HEAFG = arc HDCBG or HG is a diameter of the circle circumscribing the quadrilateral ABCD. Now arc AG = arc AF + arc FG = arc CB + arc BG = arc CG. Similarly arc AH = arc CH.

So $AB . BC = AG^2 - GB^2$ and $CD.DA = AH^2 - HD^2$ (by lemma II)

Hence AB . BC + CD . DA =

$AG^2 + AH^2 - HD^2 - GB^2 = HG^2 - HD^2 - GB^2$,

because HG is a diameter of the circumcircle and by proposition I).

Again $HG^2 = HD^2 + DG^2$; so $HG^2 - HD^2 = DG^2$.

Hence $AB . BC + CD . DA = DG^2 - GB^2 = DF . DB$ (by Lemma I).

DF = chord of (arc DA + arc AF)

= chord of (arc DA + arc CB) , since arc AF = arc CB,

= chord of (arc EC + arc CB) = EB.

DF and EB are called the third diagonals of the cyclic quadrilateral ABCD. (Note that we got them on replacing the side CB or CD by AF or AE)

We have proved that AB . BC + CD . DA = DB . DF.

Similarly AB . DA + BC . CD = DF . AC.

From the quadrilateral AFCD, we have

CF . CD + AF . DA = AC . BD = AB . CD + BC . DA.

Let us denote the sides AB, BC, CD, And DA by the letters a, b, c, and d respectively; the diagonals AC, BD, and DF by the letters x, y, and z respectively. Then the three equations, derived above, become

$$a.\,b + c.\,d = y\,.\,z \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(i)}$$

$$a.d + b.\,c = z\,.\,x \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(ii)}$$

$$a.c + b\,.\,d = x.\,y \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(iii)}$$

(The four sides of a quadrilateral taken two at a time give only six products, namely : a . b, a . c, a . d, b . c, b . d and c . d. These products can be taken in pairs without any term repeating in any pair, only in three ways. Hence there cannot be a fourth diagonal for a quadrilateral.)

Multiplying the equations (i), (ii) , and (iii) , taken two at a time and dividing by the third, we get the squares of the diagonals. Thus

$$x = \sqrt{\frac{(ac + bd)\,(ad + bc)}{(ab + cd)}}\;,\quad y = \sqrt{\frac{(ab + cd)\,(ac + bd)}{(ad + bc)}}$$

$$\text{and} \quad z = \sqrt{\frac{(ab + cd)\,(ad + bc)}{(ac + bd)}}$$

(b) Area of the cyclic quadrilateral (in terms of its diagonals)

Consider the triangle DBF (Figure - xii), Let DO be the perpendicular from D to BF. Then

$$DO = \frac{DF\,.\,DB}{\text{diameter}} \quad \text{(by proposition III)}$$

Now the quadrilateral ABCD can be considered as made up of the two triangles ACD and ABC.

Since arc AF = arc CB, the longer arc AC and arc FB have the same midpoint G; further AC and FB are parellel.

Drop perpendiculars from B and D to the common base AC to meet it at N and M. Produce DM to meet FB at O.

Then MO = BN. Hence DM + NB = DM + MO = DO

Area of the quadrilateral ABCD = area of the triangle ADC + area of the triangle ABC

= (1 / 2) AC . DM + (1 / 2) AC . NB

= (1 / 2) AC (DM + NB) = (1 / 2) AC . DO.

Using the value of DO, derived above, we get

$$\text{the area of the quadrilateral ABCD} = \frac{AC \,.\, BD \,.\, DF}{2 \text{ diameter}} = \frac{x \,.\, y \,.\, z}{4.\ R}$$

(c) *Area of a cyclic Quadrilateral (in terms of its sides).*

Let ABCD be a cyclic quadrilateral; the diagonal BD divides the quadrilateral into two triangles ABD and CBD (Figure - xiii a) . AE and CF are altitudes from A and C to the common base BD. M is the mid-point of BD. Further we denote by R, the radius of the circle circumscribing the quadrilateral ABCD. Now,

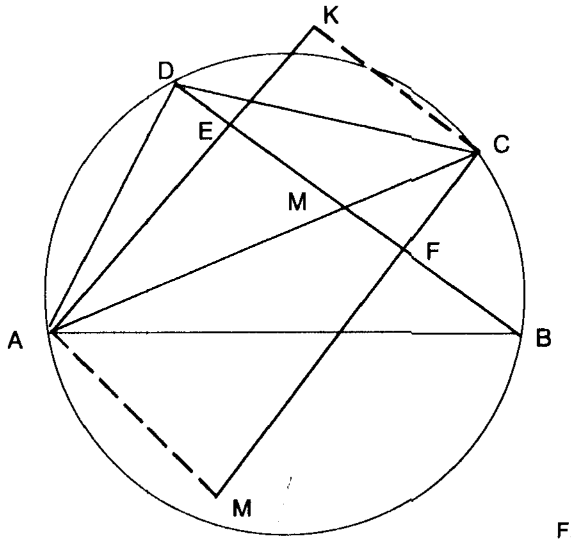


Figure - xiii a

$$AE = \frac{AD \cdot AB}{2R} \text{ and } CF = \frac{CD \cdot CB}{2R} \text{ (by proposition III)}$$

Area of the triangle ABD

$$= \frac{1}{2} \frac{BD \cdot AD \cdot AB}{2R} \qquad \text{and}$$

Area of the triangle CBD

$$= \frac{1}{2} \frac{BD \cdot CD \cdot CB}{2R}$$

AE is parallel to CF because both are perpendicular to a common line, viz. BD.

Produce AE to K so that EK = FC. Similarly produce CF to H so that FH = AE. Now AK = CH = the sum of the altitudes, and AH = CK = EF = the distance between the (feet of the) altitudes. Further AC is a diagonal of both the quadrilateral ABCD and the rectangle AHCK. So

$AC^2 = AH^2 + HC^2$ and area of the quadrilateral ABCD

$= (1 / 2)\ BD \cdot HC$.

We proceed to find AH, the distance between the feet of the altitudes AE and CF. The foot of the altitude of a triangle is always closer to the smaller side; this leads to two possibilities: the mid-point of BD lies either between the feet of the altitudes or not. Let us consider the two cases :

Case I : (Figure - xiii a) Here AD < AB and BC < CD. Note that the midpoint of BD lies between E and F, the feet of the altitudes.

In the triangle BCD, the difference between the projections of CD and BC on BD = FD - BF

= (FM + MD) - (BM - FM) = 2 FM, since MD = BM.

Similarly the difference between the projections of AB and AD on BD = BE - ED = (BM + ME) - (MD - ME) = 2 ME. i.e.

$$FM = \frac{FD - BF}{2} \quad \text{and} \quad ME = \frac{BE - ED}{2}$$

$$FE = FM + ME = \frac{FD - BF}{2} + \frac{BE - ED}{2}$$, which by lemma I

$$= \frac{CD^2 - CB^2}{2 . BD} + \frac{AB^2 - AD^2}{2 . BD} = \frac{(AB^2 + CD^2) - (BC^2 + AD^2)}{2 . BD}$$

Case II: (Figure - xiii b) Here AD < AB and CD < CB. Note that the mid point of BD lies on the same side of E and F. The difference between the projections of BC and CD on BD = BF - FD = (BM + MF) - (MD - MF) = 2 . MF.

Similarly, difference of projections of AB and AD or BD = BD = BE - ED = BM + ME - (MD - ME) = 2MF

$$\text{So } MF = \frac{BF - FD}{2} \quad \text{and} \quad ME = \frac{BE - ED}{2}$$

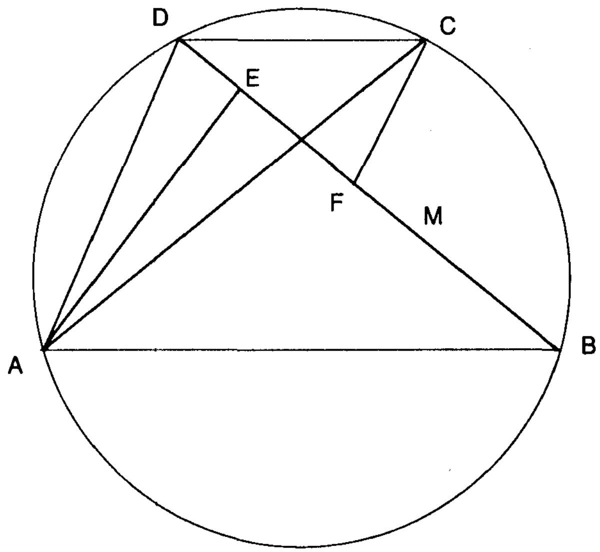


Figure - xiii b

Hence EF = ME - MF =

$$\frac{AB^2 - AD^2}{2.\,BD} - \frac{CB^2 - CD^2}{2\,.\,BD} = \frac{(AB^2 + CD^2) - (BC^2 + CD^2)}{2\,.\,BD}$$

In both the cases, difference of the projections or distance between the feet of the altitudes = Difference of the sums of the squares of pairs of opposite sides divided by twice the diagonal having the feet of the altitudes on it.

Using the abbreviations a, b, c, d for the sides of the quadrilateral and x, y for its diagonals, we can re-write the above result as

$$EF = \frac{(a^2 + c^2) - (b^2 + d^2)}{2\,.\,y}$$

$$(\text{sum of the altitudes})^2 = x^2 - \frac{[(a^2 + c^2) - (b^2 + d^2)]^2}{4\,.\,y^2}$$

$$(\text{area of the quadrilateral ABCD})^2$$

$$\left[\, x^2 - \frac{[\,(a^2 + c^2) - (b^2 + d^2)\,]^2}{4\,.\,y^2} \,\right] \frac{y^2}{4}$$

$$\frac{4.\,x^2.y^2 - [\,(a^2 + c^2) - (b^2 + d^2)\,]^2}{16}$$

We have proved earlier that

$$x^2 = \frac{(ad + bc)\,(ac + bd)}{(ab + cd)} \quad \text{and} \quad y^2 = \frac{(ab + cd)\,(ac + bd)}{(ad + bc)}$$

So $x^2 . y^2 = (ac + bd)^2$

Hence (area of the quadrilateral)^2

$$= \left[\frac{ac + bd}{2}\right]^2 - \left\{\left[\frac{a}{2}\right]^2 + \left[\frac{c}{2}\right]^2 - \left[\frac{b}{2}\right]^2 - \left[\frac{d}{2}\right]^2\right\}^2$$

$$= \left\{\left[\frac{ac + bd}{2}\right] + \left[\frac{a}{2}\right]^2 + \left[\frac{c}{2}\right]^2 - \left[\frac{b}{2}\right]^2 - \left[\frac{d}{2}\right]^2\right\}^2$$

$$\times \left\{\frac{ac + bd}{2} - \left[\frac{a}{2}\right]^2 - \left[\frac{c}{2}\right]^2 + \left[\frac{b}{2}\right]^2 + \left[\frac{d}{2}\right]^2\right\}$$

$$= \left\{\left[\frac{a}{2} + \frac{c}{2}\right]^2 + \frac{bd}{2} - \left[\frac{b}{2}\right]^2 - \left[\frac{d}{2}\right]^2\right\}$$

$$\times \left\{\left[\frac{b}{2} + \frac{d}{2}\right]^2 + \frac{ac}{2} - \left[\frac{a}{2}\right]^2 - \left[\frac{c}{2}\right]^2\right\}$$

Since $p^2 + q^2 = 2pq + (p - q)^2$, we have $p^2 + q^2 > 2pq$.

Hence the last three terms in the two factors of the product above, give negative values; so we write them as

$$= \left\{\left[\frac{a}{2}+\frac{c}{2}\right]^2 - \left[\frac{b}{2}-\frac{d}{2}\right]^2\right\} \times \left\{\left[\frac{b}{2}+\frac{d}{2}\right]^2 - \left[\frac{a}{2}-\frac{c}{2}\right]^2\right\}$$

$$\left[\frac{a}{2}+\frac{c}{2}+\frac{b}{2}-\frac{d}{2}\right]\left[\frac{a}{2}+\frac{c}{2}-\frac{b}{2}+\frac{d}{2}\right]$$

$$\left[\frac{b}{2}+\frac{d}{2}+\frac{a}{2}-\frac{c}{2}\right]\left[\frac{b}{2}+\frac{d}{2}-\frac{a}{2}+\frac{c}{2}\right]$$

Denoting $\frac{a}{2}+\frac{b}{2}+\frac{c}{2}+\frac{d}{2}$ by s, we get

$(\text{Area of the quadrilateral ABCD})^2 = (s-a)(s-b)(s-c)(s-d)$

or area $= \sqrt{(s-a)\,(s-b)\,(s-c)\,(s-d)}$

Formula for radius of the circle circumscribing the cyclic quadrilateral

Sum of the altitudes = AE + CF = $\frac{ad}{2R} + \frac{bc}{2R}$ (by proposition III)

Area of the cyclic quadrilateral ABCD $= \frac{1}{2} \quad \frac{ad + bc}{2R}$

Hence $R = \dfrac{y\,.\,(ad + bc)}{4.\ \text{area of the quadrilateral ABCD}}$

$$= \frac{1}{4}\sqrt{\frac{(ab + cd)\,(ac + bd)}{(ad + bc)}} \times \frac{(ad + bc)}{\text{area}}$$

$$= \frac{1}{4}\sqrt{\frac{(ab + cd)\,(ac + bd)\,(ad + bc)}{(s-a)\,(s-b)\,(s-c)\,(s-d)}}$$

NILAKANTHA's GEOMETRICAL DEMONSTRATIONS

Nilakantha Somayaji, in his commentary on the *Āryabhatiya* demonstrates many results formulated in the text. Here we present some.

Result I :

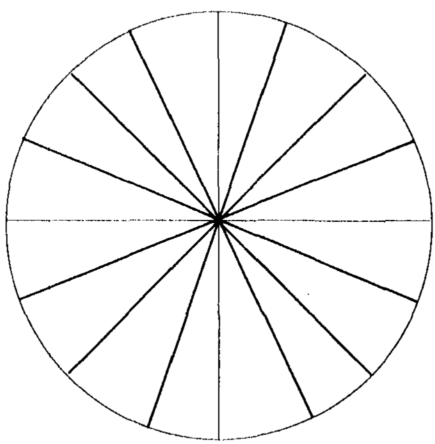
'Area of a circle = half the circumference x half the diameter'.

This result was known to the Indian Scholars from very early times; the early Jaina works such as Tattvārthadhigama-bhāsya of Umasvati (c. 150 B.C.) indicate this fact but give no hint at deriving this result. [67].

Nilakantha's demonstration is as follows:

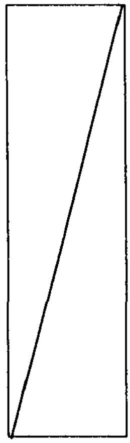
The interior of a circle is cut up into a number of tapering laminae, by means of line-segments joining the centre to different points on the circumference, as shown in the Figure-xiv. If the number of such laminae is made sufficiently large, the base of each lamina will approximate to a straight line segment. When two of these thin

Figure-xiv



laminae are juxta-posed, as show in Figure-xv, a rectangle with one side equal to the radius and the adjacent side equal to the base of a lamina results. The set of laminae, which cover the complete interior of the circle can thus be re-arranged as very thin rectangular plates. These plates can be placed contiguously with the longer sides of consecutive plates touching each other, (through-out their lengths). The result is a rectangular sheet, whose adjacent sides are half the circumference and half the diameter.

Figure - xv



Thus the area of the circle = area of the rectangular sheet)

= (1/2) circumference x (1/2) diameter.

Result II : -

The chord of one-sixth of the circumference of a circle is equal to half the diameter.

Proof : -

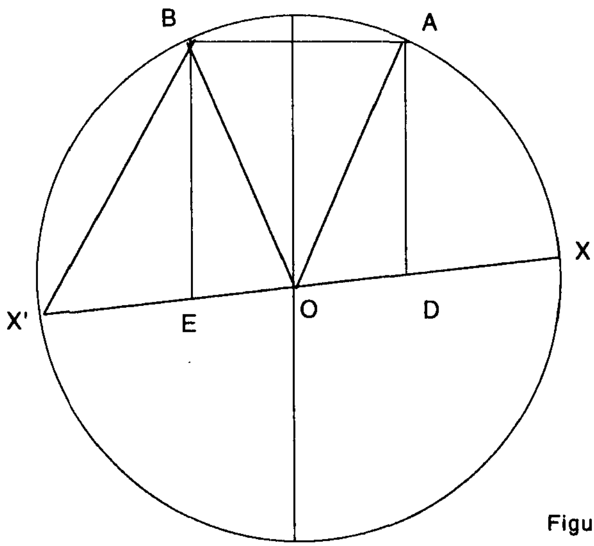


Figure-xvi

Let X'OX be a diameter of a circle with center O and radius OX. (Figure - xvi). Mark a point A on the circle so that the chord XA = OX. Join OA and XA. Now the triangle OXA is equilateral, with OX as a side. In a same manner, construct an equilateral triangle having OX' as a side; let B the vertex of this triangle. Let the perpendiculars drawn from A and B to X' OX meet it at the points D and E respectively.

Since OAX is an equilateral triangle, D is the mid-point of OX. Just so E is the mid-point of X'O.

Now the chord AB = ED = (1/2)X'O + (1/2) OX = (1/2) X'OX

= (1/2) the diameter

Result III : -

Approximation of an arc (of a circle) in terms of its sine and versine. The length of the arc is approximately equal to the square-root of the sum of one and one-third the square of its versine and the square of its sine. (recall that the versine = the radius minus the cosine).

The diameter BK of the circle (Figure-xvii) is the perpendicular bisector of the chord AX. So the arc AB = (1/2) the arc ABX. Let the arc AB = a, the sine of arc AB = AP = b, and the versine of arc AB = PB = s. Then

$$a \sim \sqrt{b^2 + \frac{4}{3} . s^2} ;$$

Further let us denote : -

1) the arcs AC, AD .... by a1, a2 .... respectively, where C, D, ... are the midpoints of arcs AB , AC, ... respectively.

2) the chords AB, AC, AD, ... by c, c1, c2 respectively, and

3) the versines of arcs AB , AC, AD .... by s, s1, s2 .... resp.

Now $c^2 = AB^2 = AP^2 + PB^2 = b^2 + s^2$ .... (i)

If the line joining C to O (the center of the circle) intersects the chord AB at Q, then

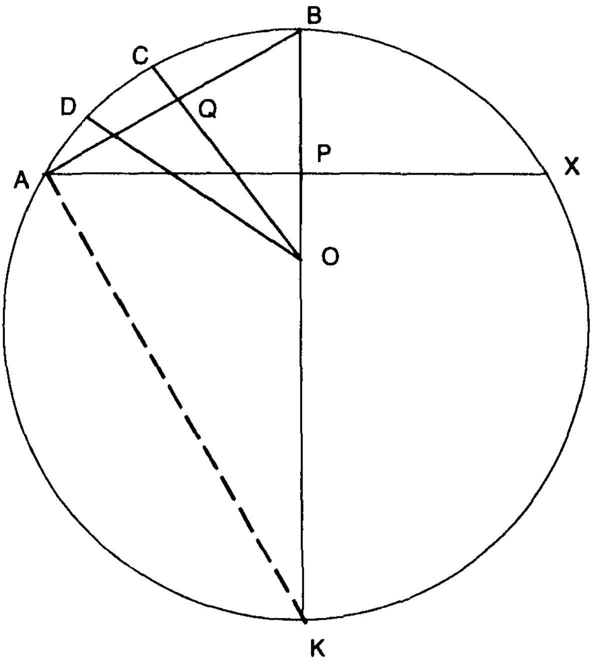


Figure - xvii

$$c_1^2 = AC^2 = AQ^2 + QC^2 = [\frac{1}{2}.C]^2 + S_1^2 = \frac{1}{4}C^2 + S_1^2$$

Similarly,

$$c_2^2 = [\frac{1}{2} c_1]^2 + s_2^2 = \frac{1}{4}[\frac{1}{4} c^2 + s_1^2] + s_2^2 = \frac{1}{4^2}.c^2 + \frac{1}{4} s_1^2 + s_2^2$$

$$c_3^2 = \left[\frac{1}{2} \cdot c_2\right]^2 + s_3^2 = \frac{1}{4^3} \cdot c^2 + \frac{1}{4^2} \cdot s_1^2 + \frac{1}{4} \cdot s_2^2 + s_3^2$$

— — — — — — — — — — — — — — — — — — — —

— —

$$c_n^2 = \frac{1}{4^n} \cdot c^2 + \frac{1}{4^{n-1}} \cdot s_1^2 + \frac{1}{4^{n-2}} \cdot s_2^2 + \ldots\ldots \frac{1}{4} s_{n-1}^2 + s_n^2$$

When the arc is very small, its chord approximates to the arc; so we can write

$$a_n^2 \sim \frac{1}{4^n} c^2 + \frac{1}{4^{n-1}} \cdot s_1^2 + \frac{1}{4^{n-2}} \cdot s_2^2 + \ldots\ldots\ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + \frac{1}{4} \cdot s_{n-1}^2 + s_n^2$$

Since

$$a_n = \frac{1}{2} a_{n-1} = \frac{1}{2^2} \cdot a_{n-2} = \ldots\ldots\ldots\ldots\ldots\ldots = \frac{1}{2^n} . a$$

We have

$$a^2 \sim c^2 + 4 \cdot s_1^2 + 4^2 \cdot s_2^2 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + 4^n s_n^2 \ldots\ldots\ldots\ldots \text{(ii)}$$

From the similar triangles ABP and KBA we have

$BP = s = \dfrac{AB^2}{BK} = \dfrac{c^2}{d}$, where BK = d, the diameter of the circle.

$$CQ = s_1 = \frac{AC^2}{d} = \frac{AQ^2 + CQ^2}{d} = \frac{1}{d}\left[\frac{AB^2}{4} + CQ^2\right] =$$

$$\frac{1}{d}\left[\frac{1}{4}.c^2 + s_1^2\right] \sim \frac{1}{4d}\, c$$

$\sim \frac{1}{4}.s$ (since s_1 is small and $c^2 = s.d$).

Similary $s_2 \sim \frac{1}{4}.s_1 = \frac{1}{4^2}.s$; $s_3 \sim \frac{1}{4^3}.s$ and so on.

Hence

$$s^2 + 4.s_1^2 + 4^2.s_2^2 + \ldots\ldots\ldots \sim s^2\left[1 + \frac{1}{4} + \frac{1}{4^2} + \ldots\ldots\ldots\right]$$

$$= \frac{4}{3}.s^2 \quad \ldots \text{(iii)}$$

Application of (i) and (iii) reduces (ii) to

$$a^2 \sim b^2 + \frac{4}{3}.s^2$$

Substituting a = R . x, b = R sin x and s = R (1 - cos x), we get :

$$x \sim \sqrt{\sin^2 x + \frac{4}{3}(1 - \cos x)^2} = \sqrt{\frac{5}{2} + \frac{1}{6}(\cos 2x - 16 \cos x)}$$

The right hand side, on simiplification, reduces to

$$x - \frac{x^5}{180}$$

So Nilakantha's formula gives the result correct to the fourth power.

In the above discussion Nilakantha has used the formula for the sum of an infinite geometrical series. He has stated the general rule thus:

The sum of an infinite series, whose terms (from the second onwards) are obtained by dividing the preceding ones by the same divisor (viz., the denominator of the first term), is always equal to the numerator of the first term divided by one less the common divisor.

Nilakantha demonstrates the validity of this statement in a particular case, namely the series

$\frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \ldots\ldots\ldots\ldots$ which occurs in the discussion above.

Now

$$\frac{1}{3} - \text{the first term} = \frac{1}{3} - \frac{1}{4} = \frac{1}{4.3}$$

$$\frac{1}{4.3} - \text{the second term} = \frac{1}{4.3} - \frac{1}{4.4} = \frac{1}{4.4.3}$$

$$\frac{1}{4.4.3} - \text{the third term} = \frac{1}{4.4.3} - \frac{1}{4.4.4} = \frac{1}{4.4.4.3}$$

and so on. Hence

$$\frac{1}{3} - (\text{sum of the first n terms of the series}) = \frac{1}{4^n \cdot 3}$$

When n is very large the right hand side becomes negligibly small. Hence the sum of the infinite series = (1 / 3). This method can be applied to any series of the type envisaged in Nilakantha's statement.

The enunciation of the general formula for the sum of an infinite convergent geometrical series and its demonstration in particular cases seem to be Nilakantha's original contribution.

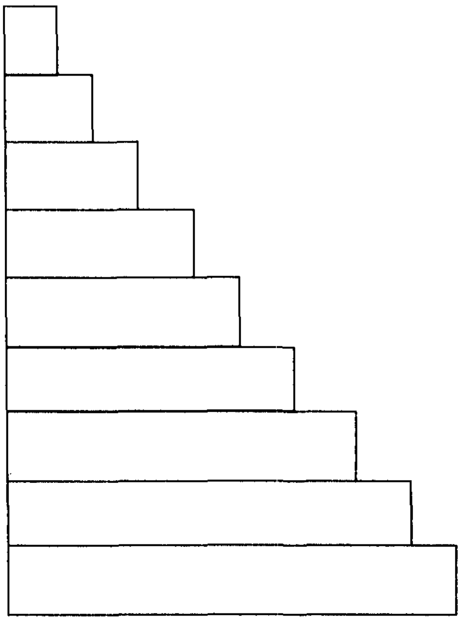
Next we come to the geometrical methods of the summation of series : -

Result IV : - The arithmetical series.

Proof : Each term of an arithmetical series can be represented by a rectangular strip of length equal to the number denoted by the term and of width one unit. These strips arranged one after another in a contiguous manner forms a lamina as indicated in the Figure-xviii; the lamina represents the series and the area of the figure the sum of the arithmetical series.

Two such laminae are put together, the first one as in the Figure - xviii and the second one in an inverted position as in the Figure-xix. The adjacent sides, of the rectangular lamina obtained, are n units and (a+t) units, where n is the number of terms and a and t are the first and last terms respectively of the arithmetical series. The area of the rectangular lamina will be n . (a + t). So the sum of the arithmetical series or the area of the Figure - xviii

Figure xviii



$$= \frac{1}{2} n.(a+t)$$

Hence the sum of the first n natural numbers = $n\frac{1}{2}.(n+1)$.

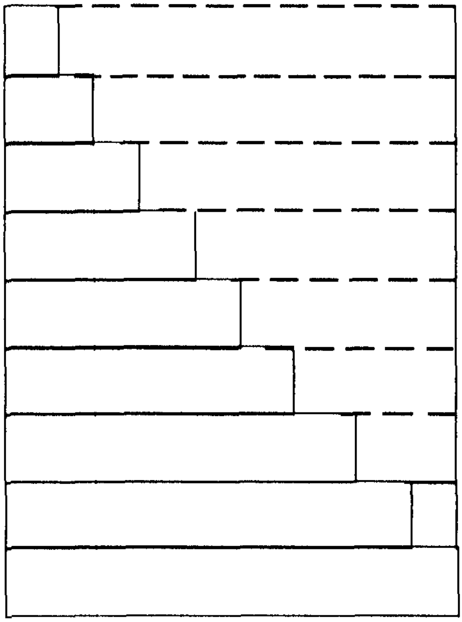


Figure - xix

Result V : - A series of triangular numbers i.e.,

$$\sum_{r=1}^{n} \frac{r.(r+1)}{2} = \frac{n.(n+1).(n+2)}{6}$$

Proof : - (We assume all the slabs or tiles occurring hereafter to be of thickness one unit; the shape of the top (or bottom) face alone will be specified.) Take 6 slabs each in the shape of Figure-xviii, representing S_n, the sum of the first n natural numbers. Combine these slabs in pairs to form rectangular slabs, in the shape of Figure-xix, having the adjacent sides n and (n+1) units. Now place one of

these slabs flat on the ground and the other two vertically (on the ground) touching the sides of the first one, so that the three slabs cover (on the ground) a rectangular area of adjacent sides (n+1) and (n+2) units. Now the three slabs form the bottom and two side-walls of a semi-open cuboid or rectangular parallelopiped of (outer) edges n ; (n+1) and (n+2) units.

Now repeat the procedure with the 6 slabs, each of which represents S_{n-1} , the sum of the first (n-1) natural numbers; the three rectangular slabs are placed (in the same manner) inside the open cuboid already formed. Continue the procedure until the 6 slabs representing S_2, the sum of the first 2 natural numbers have been stacked. The space left, in the open cuboid can be filled by the 6 slabs representing S_1. Thus a solid cuboid whose co-terminus edges are n, (n+1), and (n+2) is formed. So

$$6 \cdot \sum_{1}^{n} \frac{r.(r+1)}{2} = n.(n+1).(n+2) \qquad \text{or}$$

$$\sum_{1}^{n} \frac{r.(r+1)}{2} = \frac{n.(n+1).(n+2)}{6}$$

Result VI : -

Similarly 6 sets each, of square slabs of sides n, n-1, n-2 can be arranged to produce a (solid) cuboid whose co-terminus edges are n, (n+1) and (2n +1). So

$$\sum_{r=1}^{n} r^2 = \frac{n.(n+1).(n+2)}{6}$$

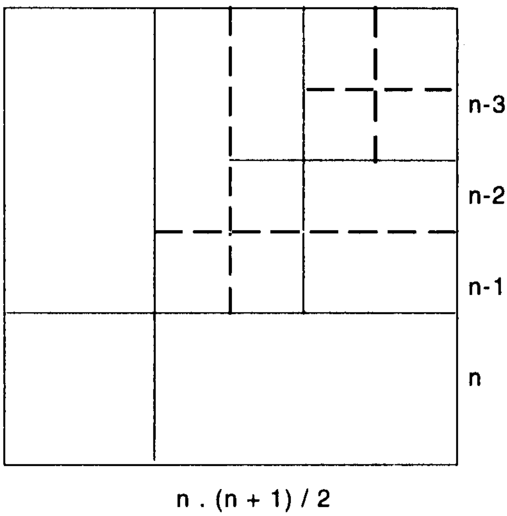
Result VII : -

$$\sum_{r=1}^{n} r^3 = \left[\frac{n.(n+1)}{2}\right]^2$$

For the demonstration of this result our starting point is the right-hand-side. A square slab of side $\frac{n \cdot (n+1)}{2}$ units (and thickness one unit) is cut up into L-shaped slabs of width n, n-1, n-2, ...1 units as in the Figure - xx. These slabs can be made to yield n, n-1, n-2 ..... 1 square-slabs of sides n, n-1, n-2 .... + 1 units, respectively (and thickness one unit), as explained below. Note that each arm of the L-shaped slab of width n units is a rectangular slab, which can be sub-divided into (n-1) / 2 square slabs of side n units in addition to a square slab of side n at the corner. So the number of n-square slabs =

$$2 . \frac{n-1}{2} + 1 = n.$$

Figure - xx



Now if the slabs of equal areas are piled up, one above the other, solid cubes of edges n, n-1, n-2 , ... units respectively will result. Hence

$$1^3 + 2^3 + 3^3 + \ldots\ldots\ldots\ldots + n^3 = \left\{ \frac{n . (n+1)}{2} \right\}^2$$

Nilakantha does not claim any originality for these proofs. It is not unlikely that these proofs were known to some members of the Āryabhatan school earlier.

MISCELLANEOUS RESULTS

I . *Āryabhata Vindicated*

Aryabhata's *Āryabhatiya* is a very concise work. Several of its verses are very succinct and difficult to understand. Some verses lend themselves to more than one interpretation. This is not unusual in Sanskrit poems. Here we examine two verses of the *Aryabhatiya*.

The first one is verse 6 of Ganita-pada (i.e., second part of the *Āryabhatiya*). It runs as follows :

"Tribhujasya phalasarira samadalakotibhujardhasamvargah

urdhvabhujatatsamvargardham sa ghanah sadasrir iti"

This verse was first interpreted as " Half *(ardha)* the product *(samvarga)* of the base *(bhuja)* and the height *(koti)* of an equilateral *(samadala)* triangle *(tribhuja)* forms the area *(phala)* of the solid *(sariram)* formed. Half *(ardha)* the product *(samvarga)* of this (area) and the perpendicular *(urdhvabhuja)* or height is the volume (ghana) of the solid with six edges *(sadasri)* pronounced *SHADASHRI*) or triangular pyramid." This interpretation gained currency and Āryabhata was accused of giving an incorrect formula for the volume of the triangular pyramid.

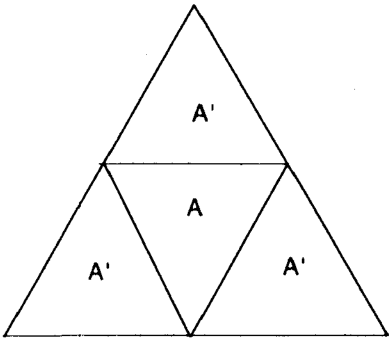
Kurt Elfering adopting some ideas of Conrad Muller (who asserted in 1940 that Aryabhata's stanzas contained the correct results in a hidden way) has given the following interpretation giving the correct formula: " Half the product of the base and the height of an equilateral triangle (forms) a solid by (its) area. Half the product of this (area) and the perpendicular (of the solid, i.e., the height of the pyramid) forms a rectangular solid (whose volume is equal to) *six* (sad) pyramids (*asri*), so it is explicitly stated (=iti)" [14]

(Note the different meanings of *sadasri* - in the first case it is

taken as a composite word and interpreted as 'six-edged solid' while in the second the word has been split up as *sad*, meaning six and *asri* whose precise meaning is 'top, peak, edge' here means "pyramid" or edged (rectangular solid).

The triangular pyramid according to Elfering, is made by dividing an equilateral trigangle and folding up the 3 peripheral triangles A'. (Figure - xxi) over the central one, so that the total surface-area of the pyramid = area of the original triangles. Six such pyramids together will have a volume equal to half the product of the original triangle and the height of the pyramid.

Figure xx1



The base area of such a pyramid = (1 / 4) area of the original triangle; so the volume of 6 such pyramids = 6 x (1/3) area of the base x height

$$= 6 \times \frac{1}{3} \; \frac{\text{area of the original triangle}}{4} \; \times \text{height}$$

= (1 / 2) area of the original triangle x height.

The interpretation is no doubt ingenious. It is surprising that it did not occur to the able mathematicians of the Āryabhatan school. Brahmagupta, a vehement critic of Āryabhata, knew the correct formula for the volume of a pyramid; he did not attack this verse. This lends plausibility to Elfering's interpretation given above. Further the

mistaken formula for the volume of the pyramid is incompatible with Āryabhata's knowledge of the correct number of balls in a pile in the shape of pyramid on a triangular base (given in verse 21 of Ganitapāda).

The second one is verse 7 of Ganita-pada, which is;

"Samaparināhasyārdham viskambhardhahatameva vrttaphalam
tan nijamūlena hatam ghana golaphalam niravasesam."

This verse had been interpreted by the Āryabhatan school thus: "Half *(ardha)* the circumference *(of a circle)* *(sama-parinaha)* multiplied *(hata)* by half the diameter *(viskambhardha)*, that is the area of the circle *(vrttaphala);* this area i.e, of the *(great)* circle multiplied *(hatam)* by its square-root *(nija mula)* is the volume of the solid sphere *(ghana golaphala)*, i.e. , in symbols.

$$V = \Pi . R^{2} . \sqrt{\Pi . R^{2}} = \sqrt{\Pi} . \Pi R^{3}$$, which is incorrect.

Elfering gives the following translation for the verse: "Half the circumference multiplied by half the diameter, that is the area of the circle ; this one (i.e., the circumference) multiplied by its defining base (i.e., the radius) is exactly the surface of the hemisphere."

The word *nijamula* was translated as 'its square-root' by the scholiasts of the Aryabhatan school, while Elfering says 'it is necessary to translate the word *mula* by its original meaning, "base; cause"; in such a way nijamula yields 'own defining base' (for the circumference) and that can only be the radius R." A strong argument for this interpretation lies in the passages in Paramesvaran's commentary on the verse 7 and 9 of Ganita-pada. In this way the second half-stanza gives the correct geometrical result for the surface of the hemisphere, in symbols

Area of the circle $= \Pi R . R = \Pi R^{2}$

Surface-area of the hemisphere $= 2 \Pi R.R = 2 \Pi R^{2}$

Paramesvaran introduces the second half-stanza of verse 7 with the comment

ghana-samavrttaksetrasya phalam aparārdhenaha

which means: "he (Āryabhata) declares by the following half (-stanza) the result for the area of the hemisphere".

The word ghanasamavrtta can be interpreted as "hemisphere" because it denotes "uniformly circular bell".

Another strong argument is a passage in Paramesvaran's commentary on verse 9 of Ganita-pada, which deals with the calculation of geometrical areas. The text reads :

Ghanagole'pi vrttaphalasya mūlam ucchrayah.

This can be translated : "in the hemisphere the perpendicular also is the base element for the area of the circle (on which the hemisphere sits).

The word *ghana-gola* has been translated in the above meaning. The perpendicular of the hemisphere is equal to the radius R and this is denoted by *mūla*, "basic element; cause" ; and that is supported by facts, because both the area and circumference depend on R. The word *mula* appears in its original meaning as well. [14; 30].

2. Derivation of the formula $\sin(x + y) = \sin x \cdot \cos y + \cos x \cdot \sin y$

One method of deriving the formulae

$\sin(x \pm y) = \sin x \cdot \cos y \pm \cos x \cdot \sin y.$

and

$\cos(x \pm y) = \cos x \cdot \cos y \mp \sin x \cdot \sin y$

and three others for $\sin(x + y) = \sin x \cdot \cos y + \cos x \cdot \sin y$ in the *Yukti Bhasa* are given below.

Let a quadrant arc XY of a circle (Figure-xxii) be divided into 24 equal parts.. Take the measure of each part as one unit. So each unit

of arc faces an angle $3^0 45'$ or (Π / 48) radian at the center of the circle. Further we take angle YOC = x units and the angle COA = y units, where x > y. We take R as the radius of the circle. Then the arc YC = R . x and arc CA = R . y.

Method I:

On the arc CY take a point B so that arc CB = arc CA. Join A to B and C to O;

Let them intersect at M. From A, C, M and B draw perpendiculars to OX and OY.

Let them meet OX and OY at D, E, F, G and K, H, J, L resp.

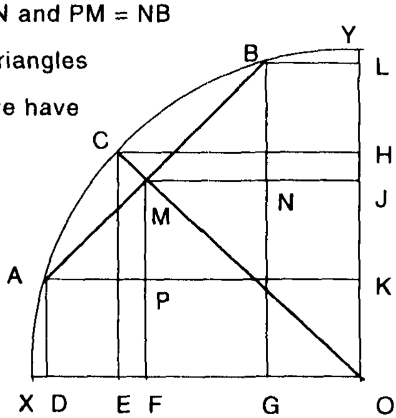
Now CH = sine of arc CY = R . sin x.

OH = EC = cosine of arc CY = R . cos x.

AM = MB = sine of arc CA = R . sin y.

OM = cosine of arc CA = R . cos y.

Figure - xxii



Note that AP = MN and PM = NB

From the similar triangles OMJ and OCH, we have

$$\frac{MJ}{CH} = \frac{OM}{OC}$$

Therefore

$$MJ = \frac{CH . OM}{OC}$$

$$= \frac{R. \sin y . R \cos x}{R}$$

From the similar triangles APM and CEO, we have

$$\frac{AP}{CE} = \frac{PM}{EO} = \frac{MA}{OC}$$ Therefore

$$AP = \frac{AM \cdot CE}{R} = \frac{R \cdot \sin y \cdot R \cos x}{R}$$ and

$$PM = \frac{EO \cdot AM}{OC} = \frac{R \cdot \sin x \cdot R \sin y}{R}$$, because EO = CH.

From the similar triangles OMF and OCE, we have

$$\frac{FM}{EC} = \frac{OM}{OC}$$ Therefore $$FM = \frac{EC \cdot OM}{OC} = \frac{R \cos x \cdot R \cos y}{R}$$

Now AK = AP + PK + = AP + MJ = MJ + AP, Which is the same as

$$R \sin (x + y) = \frac{R \sin x \cdot R \cos y + R \sin y \cdot R \cos x}{R}$$

and BL = NJ = MJ - MN = MJ - AP , which is the same as

$$R \sin (x - y) = \frac{R \sin x \cdot R \cos y - R \sin y \cdot R \cos x}{R}$$

Further, DA = FM - PM , which is the same as

$$R \cos (x + y) = \frac{R \cos x \cdot R \cos y - R \sin x \cdot R \sin y}{R}$$

and GB = GN + NB = FM + PM , which is the same as

$$R \cos (x - y) = \frac{R \cos x \cdot R \cos y + R \sin x \cdot R \sin y}{R}$$

In the *Yukti Bhāsa*, the above formulae are derived, in the manner indicated above, for the particular cases (i) x = 2, y = 1 and (ii) x = 14, y = 6 units.

The formulae derived above, reduce to

$\sin (x \pm y) = \sin x \cdot \cos y \pm \cos x \cdot \sin y$

and $\cos (x \pm y) = \cos x \cdot \cos y \mp \sin x \cdot \sin y$

Alternate methods of deriving

$$\sin (x + y) = \sin x \cdot \cos y + \cos x \cdot \sin y$$

Method II :

Let arc AC = R . x and arc BC = arc CD = R. y. (Figure-xxiii) ; P the center and EPH a diameter of the circle. DG and BF are parallels to EH. PA, the perpendicular to EH, intersects DG at K. PC intersects the chord DB at L. Now L and K are midpoints of DB and DG respectively; so we have LK = (1/2) BG. Arc BG = 2 arc AC; hence (1/2) BG = R.sinx.

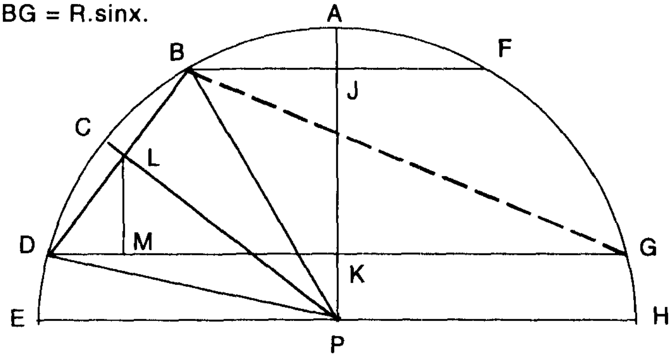


Figure - xxiii

Now DKP and DLP are right angles; so PKLD is a cyclic quadrilateral. Hence angle DPL = angle DKL. Draw LM perpendicular to DG. Then the right triangle LMK and DLP are similar. Therefore

$$\frac{LM}{DL} = \frac{LK}{DP}$$

Hence $$LM = \frac{LK \cdot DL}{DP} = \frac{(1/2)\, BG \cdot DL}{DP}$$

$$= \frac{R \sin x \cdot R \sin y}{R}$$

From the right triangle LMK, we have $MK^2 = LK^2 - LM^2 =$

$$= (R \sin x)^2 - \left\{ \frac{R \sin x \,.\, R \sin y}{R} \right\}^2$$

$$= (R \sin x)^2 \left\{ 1 - \left[\frac{R \sin y}{R} \right]^2 \right\}$$

$$= (R \sin x)^2 \left\{ \frac{R \cos y}{R} \right\}^2$$

Thus

$$MK = \frac{R \sin x \,.\, R \cos y}{R}$$

Similarly

$$DM^2 = LD^2 - LM^2 = (R \sin y)^2 - \left[\frac{R \sin x \,.\, R \sin y}{R} \right]^2$$

$$= (R \sin y)^2 - \left[\frac{R \cos x}{R} \right]^2$$

Thus

$$DM = \frac{R \sin y \,.\, R \cos x}{R}$$

DK = MK + DM . Substituting the values of MK and DM, we get

$$R \sin (x + y) = \frac{R \sin x.\, R \cos y + R \sin y \,.\, R \cos x}{R}$$

or

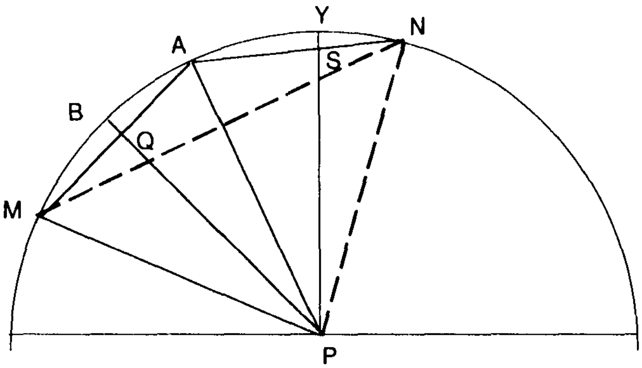
$\sin (x + y) = \sin x \,.\, \cos y + \cos x \,.\, \sin y.$

The Yukti Bhāsa, employs this method, to derive the above result, for the particular case x = 14 and y = 6 units.

Method III :

Let the arc AY = Rx and arc AB = Ry. (Figure - xxiv) ; P the center of the circle. Take the points M and N on the semi-circle, so that arc AY = arc YN and arc AB = arc BM. Draw the chords MA and AN. Join PB and PY; let them intersect the chords MA and AN at Q and S respectively.

Figure - xxiv



Q and S are the midpoints of AM and AN respectively; so QS = (1/2) MN. Now QASP is a cyclic quadrilateral (because PQA and PSA are right angles). Hence AS . PQ + QA . PS = PA . QS. (This result has been derived in the chapter 'Paramesvaran and cyclic quadrilateral') We know that AS = R sin x, PQ = R.cosy, QA = R siny, PS = R cos x, PA = R and QS = (1/2) MN = R sin (x + y),

So the above equation can be writter as:

$R \sin x \, . \, R \cos y + R \sin y \, . \, R \cos x = R \, . \, R \sin (x + y)$

i.e. , $\sin (x + y) = \sin x \, . \, \cos y + \sin y \, . \, \cos x$.

The particular case when x = 5 and y = 3 units has been worked out, by this method, in the *Yukti Bhāsa.*

Method IV;

This is a variant of method III; here the full chords are used.

ABCD is a cyclic quadrilateral inscribed in a circle. (Figure - xxv) AC , a diagonal of the quadrilateral ABCD, is also a diameter of

the circle.

Let arc BA = 2 . Rx and arc AD = 2 . Ry;

then arc BD = 2 . R (x +y)

From the cyclic quadrilateral ABCD,

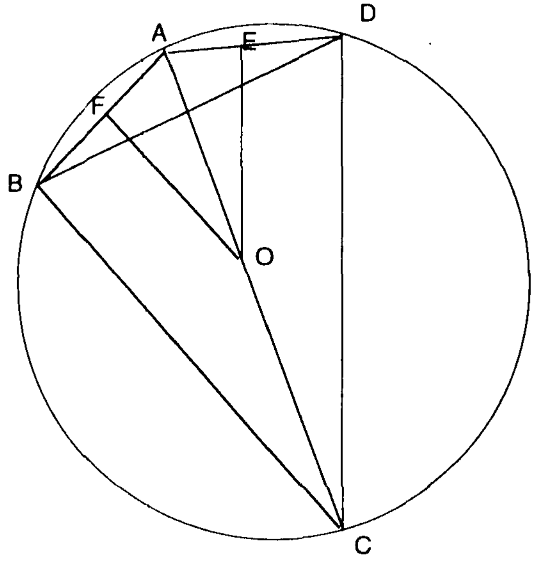


Figure-xxv

We have AC . BD = AB . CD + AD . BC Here BD, BC, CD and DA are full-chords of corresponding arcs of the circle. Hence

2R . 2 . R sin (x +y) =

2 . R sin x . 2 . R cos y + 2 . R sin y . 2 . R cos x

(Because BC = 2 OF = 2 R cos x & CD = 2 OE = 2 R cos y).

i.e., sin (x + y) = sin x . cos y + sin y . cos x.

The particular case when x = 2 and y = 1 (units) has been dealt with, by this method, in the *Yukti Bhasa.*

3. ***Surface - area of a sphere of radius R = 4 Π R*** 2

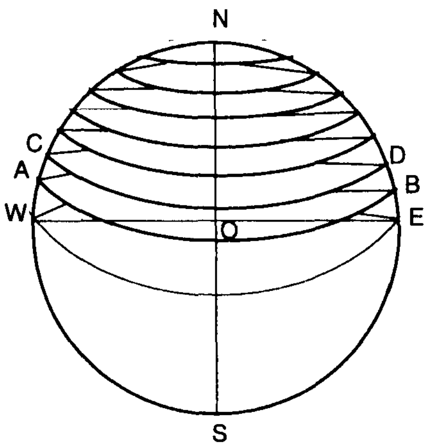


Figure - xxvi

Let O be the center and R the radius of a sphere; NWS a vertical great circle and EW a horizontal great circle of the sphere. On the sphere draw circles at constant distances apart and parallel to the great circle EW. (Figure - xxvi). The circles become smaller and smaller as we proceed from the great circle EW towards N or S. A strip or portion of the sphere between two consecutive circles, looks like a truncated cone; one rim longer than the other. If we cut the strip it opens up into a trapezoid JKTM (Figure-xxvii)

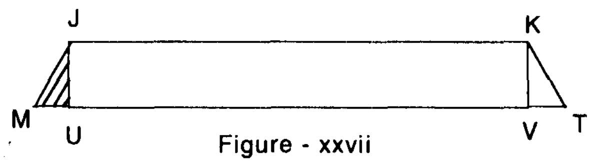


Figure - xxvii

JU and MV are perpendiculars to MT (and JK). The triangle JMU, shifted to the position of TKM' (Figure-xxviii) transforms the trapezoid JKTM to a rectangle JM'TU. Now the area of the trapezoid JKTM

$$= \frac{JK + TM}{2} . JU$$

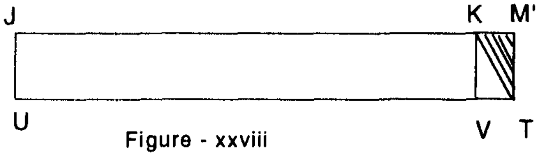


Figure - xxviii

= JM ' . JU = area of the rectangle JM ' TU.

JM' is the average of JK and TM or the circumference of the circle drawn midway between the two consecutive circles.

Hence surface-area of the hemisphere = Sum of the averages of the circumferences of the rims of such strips x h, where h is the perpendicular distance between two consecutive circles drawn midway between the consecutive circles on the hemisphere. (in ascending order of magnitudes) be p_1, p_2, p_3 .... and their respective radii B_1, B_2, B_3 ....

The radii are the sines of the arcs (of the great circle NWS) starting from the common point N and terminating at the points of intersection of the great circle NW and the circles drawn midway between the consecutive circles on the sphere. Let the circumference of a great circle of the sphere be p. Then

$$p_r = \frac{P}{R} . B_r \quad \text{for } r = 1, 2, 3 \text{ ....}$$

Now surface area of the hemisphere =

$$\sum p_r . h = \sum \frac{p}{R} . B_r . h = \frac{p . h}{R} \sum B_r$$

$$\text{Here } B_r = (b_r - b_{r+1}) . [\frac{R}{c}]^2$$

$$\text{Where } b_r = B_r - B_{r+1} \text{ for } r = 1, 2, 3, \text{ ....}$$

and c = the full chord of the arc (of the great circle NW) , lying between

two consecutive circles on the sphere. (This result has been derived as equation K, in the chapter on Mādhavan, the father of Infinitesimal Analysis.)

Surface area of the hemisphere = $\frac{p \cdot h}{R} \sum (b_r - b_{r+1}) \left[\frac{R}{c}\right]^2$

When the number n, of circles drawn on the sphere is very large

$B_n \rightarrow R$, $b_n \rightarrow 0$ and further b_1 and h approximate to c. So

Surface area of the hemisphere = $\frac{p \cdot c}{R} c \cdot \left[\frac{R}{c}\right]^2 = P \cdot R = 2 \Pi \cdot R \cdot R$

$= 2 \Pi R^2$

Hence surface area of the sphere $= 4 \Pi R^2$

4. ***Volume of a solid sphere of radius R =*** $\frac{4}{3} \Pi R^3$

Proof : We make horizontal cuts on a solid sphere so as to get slices of same thickness (Figure-xxix).

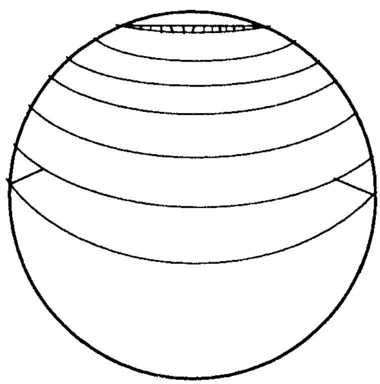
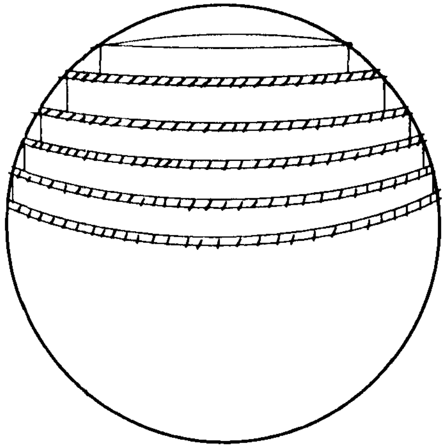


Figure - xxix

Figure - xxx



Each slice will be in the form of a truncated solid ccne; the two plane ends will not be equal in area. Draw a circle along the curved surface of the slice midway between the plane ends. The volume of the solid slice will almost be the same as that of a cylinder having its baseof the circular lamina midway between the plane ends and height equal to t, the thickness of the slice. (Fig. xxx).

Let the circumferences and radii of these circular laminas, starting from the point N be p_1, p_2, p_3 .... and B_1, B_2, B_3 ..... respectively.

These radii are the sines of the arcs of the vertical great circle.

Now $p_r = B_r . \frac{P}{R}$, where p is the circumference of the great circle and R the radius of the solid sphere.

Volume of a slice = area of the circular lamina x t

$$= \frac{1}{2} P_r . B_r . t = \frac{1}{2} . \frac{P}{R} B_r^2 . t$$

(for the derivation of the formula for the area of a circle see the chapter Nilakantha's geometrical derivations)

We have assumed the thickness of all the slices to be the same, say t. Then the distances between the point N and and centers of the various laminas will be t, 2t, 3t, ... , nt. (Figure-xxxi). We have $NG = 2R - GS = nt - GS$.

As NG takes the values 0, t, 2t, ..., (n-1)t, nt GS become nt, (n-1)t, (n-2)t, ... t, 0. Thus the various values of NG are the same as those of GS in the reverse order.

Hence $\sum NG = \sum GS$ and

$$\sum NG^2 = \sum GS^2$$

Now $NG \cdot GS = (ON - OG)(OS + OG)$

$$= R^2 - OG^2 = GF^2 = B_r^2$$

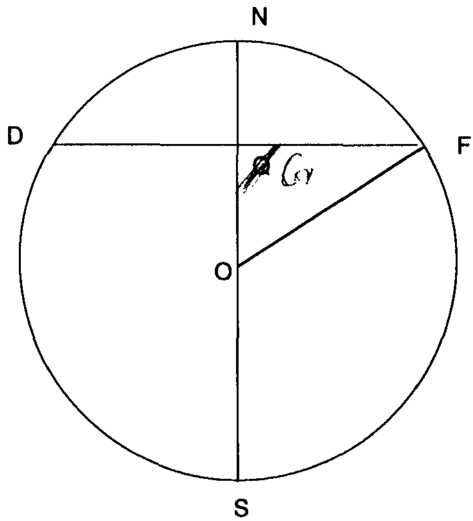


Figure - xxxi

So

$$B_r^2 = \frac{2\,NG\,.\,GS}{2} = \frac{(NG + GS)^2 - (NG^2 + GS^2)}{2}$$

$$= \frac{NS^2 - (NG^2 + GS^2)}{2}$$

$$\sum B_r^2 = \sum \frac{NS^2}{2} - \sum NG^2 = \frac{n}{2}(2R)^2 - (1^2 + 2^2 + \dots\dots + n^2)\,.\,t^2$$

Volume of the solid sphere

$$= \sum \frac{1}{2} \cdot \frac{p\,.\,t}{R} \cdot B_r^2 = \frac{1}{2} \cdot \frac{p\,.\,t}{R} \cdot \left[\frac{n}{2}(2R)^2 - (\Sigma r^2)\,.\,t^2\right]$$

$$\sim \frac{p}{2R}\left\{\frac{nt}{2}\,.\,4R^2 - \frac{n^3}{3}\,.\,t^3\right\} \text{(because } \Sigma r^2 \sim \frac{n^3}{3} \text{)}$$

$$= \frac{p}{2R}\left\{4R^3 - \frac{8R^3}{3}\right\} \text{ (because } n\,.\,t = 2R\text{)}$$

$$= \frac{4}{3}\,\Pi\,R^3 \text{ (because } p = 2\,\Pi\,R.\text{)}$$

◆

GLOSSARY

This glossary contains not only the terms occurring in this book but also some terms used by Indian mathematicians, for centuries. We have refrained from using them in this book so that the reader need not face unnecessary hurdles. The glossary will be helpful to the inquisitive reader who refers to the works listed in the bibliography.

Abja : 10^9 (the number and place)

Ādhyan (pronounced AH-DHYAN) : elite; rich and influential.

Aharganam : the number of days elapsed in an epoch.

Allopathy : therapy with remedies that produce effects differing from those of the disease treated. (compare homeopathy).

Amsa (pronounced AM-SHA) : part; same as bhāga.

Anka : number; digit.

Anka-ganitam : mathematics of numbers; arithmetic.

Antariksha : atmosphere.

Antya : 10^{15} (the number and place).

Anustup : a class of metre, used in poems, consisting of four lines of eight syllables each.

Arbuda : 10^8 (the number and place).

Ardha : middle; half.

Ardha - jya (also jya-ardha) : half chord of the double arc; corresponds to the sine.

Āsa (pronounced AH-SHA) : direction.

Āsanna (pronounced AH-SANNA) : approxiate; close to.

Āsauca (pronounced AH-SHAU-CA) : impurity; defilement caused either by child-birth (called janana-assauca) or the death of

some relation (called mrta-assauca). This used to be observed by some communities in India.

Asri (pronounced ASH-RI) : top; peak; edge.

Astrology : application of astronomy to the prediction of events; the calculation and prediction of natural phenomena; the art of judging of the reputed occult and non-physical influences of the stars and planets on human affairs.

Astronomy : the science which treats of the constitution, relative positions and motions of the heavenly bodies i.e., of all planets, satellites, stars, galaxies etc. of the universe.

Ayurveda (pronounced AH-YUR-VEDA) : a system of medical treatment which originated in India. The medicines are prepared from medicinal herbs and minerals. While under treatment, the patient has to observe strict diet restrictions.

Ayuta : 10^4 (the number and place).

Azhvanceri Tamprakkal (pronounced AH-ZH-VAH-N-CHERI TAM-PRAH-KKAL); the religious head of the namputiris of Kerala.

B

Bahu (pronounced EAH-HU) : (same as bhuja) hand; a side of a triangle or quadrilateral.

Bhāga (pronounced BHAH-GA) : an arc of a circle = (1/360) of the circle or an arc of a circle corresponding to an angle of one minute.

Bhasya (pronounced BHAH-SHYA): commentary.

Bhasya-kara (pronounced BHAH-SHYA-KAH-RA) : commentator.

Bhattacārya (pronounced BHATTA-CHAR-RYA); a title originally used along with the names of learned persons in Bengal.

Bhattatiri : a learned namputiri of Kerala.

Bhuja : same as bāhu.

Bhuja-jya : same as ardha-jya; half chord of the double arc.

Bhuja-saram (pronounced BHUJAH-SHARAM): radius minus cosine (or kotijya) of the arc.

Bhūmi : Earth; a side of a triangle or quadrilateral, taken for reference.

Bhūta : element; part; component. Ancient Hindus believed that the fundamental elements were five in number; they were earth, water, fire, air and space. So this word denoted the number five in the Bhuta Samkhya system.

Bhuta Samkhya System : a system of numeration in which each number is indicated by some well known object or concept having as many parts or components as the number it connotes.

Bija-ganita : algebra

Brahmarshi : a person who was born a brahmin and later became a sage.

C

Chandra : moon

Candra-vākhya (pronounced CHANDRA-VAH-KYA) : statements in the Katapayadi system of numeration giving the positions of the moon-rise on a number of consecutive days.

Capa (pronounced CHAH-PA) : arc of a circle.

Cara (pronounced CHA-RA) : motion.

Cera; an alternate name for Kerala.

Chola: one of the three ancient kingdoms of south India, the other two being Cera and Pandya.

D

Dala : half; a side

Danta : tooth

Dasa (pronounced DA-SHA) : 10 (both the number and the place).

Dhanus : arc of a circle.

Dik : direction.

Dis or Disa (pronounced DISH or DISHA): direction.

Dravida : old name of south India; also the name of a family of languages including Kanarese, Malayalam, Tamil and Telugu, spoken in south India.

Dravidians : original inhabitants of south India.

Deccan : the name of the region of south India, south of the Vindhya hills.

Drg-ganitam : a scheme of astronomical calculations set forth by Parames varan namputiri in 1431 A.D; also the name of the work which expounds the system known as drk.

E

Ecliptic : a great circle of the celestial sphere which is the apparent orbit of the sun. So called because eclipses can happen only when the moon is on or very near this line. Sometimes put for the plane of the ecliptic.

Eka ; one ; unit ; unit's place.

Emprantiri (pronounced EM-PRAH-N-TIRI) : a subcaste of brahmins; they work as priests at the temples.

G

Ganitam : mathematics; calculation; computation.

Gargya : a son of the sage Visvamitra.

Ghana : cube (of a number); a solid; volume.

Ghana-golam : solid sphere.

Ghana-mula : cube root.

Ghana-phalam : volume.

Ghatam (pronounced GHAH-TAM) : product; power; mutiplication.

Go ; cow.

Golam : sphere.

Gola-vid : an adept in spherics.

Gotra : (family) lineage; exogamous sects; a brahmin in his self - introduction and daily worship mentions not only his name and the name of the founder of his gotra (lineage) but also the names of certain other sages who are believed to be the remote ancestors of his family. The number of prime progenitors of gotras has been accepted as SEVEN by almost all people in India; but there are some differences in their names; while the south Indians list Agastya, Angiras, Atri, Bhrgu, Kasyapa, Vasistha and Viswamitra, the north Indians list Atri, Bharadwaja, Bhrgu, Gautama, Kasyapa, Vasistha and Viswamitra. Any sage who contributed significantly to the credit of his family-line was called a pravara. With the advent of pravaras, family-lines developed branches and individuals belonging to any branch, of course, mention the names of the founder as well as the pravaras in his family-line in the self introduction and daily worship. The total number of gotras, now, is over 500. The gotras are important for marriage alliances; a boy and a girl of the same gotra are forbidden to marry.

Graha : planet

Grahana : eclipse.

Grandha : common name for books and manuscripts.

Grandha-pura : store house of grandhas. i.e., books and manuscripts.

Guna : Multiple.

Gunana : multiplication.

Gurukula Vidyabhyasam : the system of education that prevailed in

ancient India. In this system the training in all its aspects, of the pupils took place at the residence of the teacher (guru). The pupils were, however, expected to treat their teacher with utmost reverence, ministering to his needs and obeying his commands implicitly. Not only academics like the vedas and vedangas but also martial arts such as archery etc. were included in the training.

H

Hara : divisor

Harana : division.

Hata : multiply

Hita : liking

Homeopathy : a system of medical practice founded by Hahnemann of Leipsic about 1796, according to which diseases are treated by the administration (usually in very small doses) of drugs which would produce in a healthy person symptoms closely resembling those of the disease treated. The fundamental doctrine of homeopathy is expressed in the Latin adage 'Similia similibus curantur', i.e. 'likes are cured by likes'. The traditional treatment that existed in western countries before the advent of homeopathy is called allopathy.

Horoscope : observation of the sky, the positions of planets etc. at a specific moment like the birth of a person.

I

Ili ; same as kala; an 'ili of arc' faces an angle of one minute at the center of the circle.

Illam : the name of a traditional house of a namputiri.

J

Jaladhi : 10^{14} (the number and place).

Jiva : same as jya

Jive-paraspara-nyaya : the rule for the expansion of sin (A ± B).

Jya : same as ardha - jya or jya-ardha; semi-chord of the double arc; ordinate of half arc; sine of half arc.

Jya-ardha : same as ardha - jya or jya.

Jya - samasta : same as samasta jya; full chord.

Jyas : combined name for bhuja-jya and koti-jya i.e., sine and cosine of an arc.

Jyotis : a heavenly body ; a luminous body.

Jyotis-sastra (pronounced JYOTIS-SHAH-STRA) : science of celestial bodies. It consists of two parts (i) the theoretical part, known as astronomy and (ii) the predictional part, known as astrology.

K

Kala : an arc of a circle facing an angle of one minute at the center.

Kāla (pronounced KAH-LA) : time (past, present and future).

Kali era or Kali yugam : name of the current era, according to the south Indians; it began on Friday the 18th of February 3102 B.C.

Kalpa : a period of 432×10^7 solar years; this period is called an aeon (see mahayuga and manu).

Karana : half a tithi or half a lunar day; process; working.

Karana text : working manual.

Karna : ear; diagonal of a quadrilateral; hypotenuse of a right triangle.

Katapayadi (pronounced KATA-PA-YAH-DI) : a system of numeration where the letters of the Sanskrit alphabet are used to denote the digits.

Kathakali : a form of silent dance drama peculiar to Kerala.

Kendra : center (of a circle).

Kha : sky.

Kha golam : celestial sphere.

Kharva : 10^{10} (the number and place).

Kolla Varsham : same as Malabar era; an era followed in Kerala state; it commenced in 825 A.D.

Koti : height; 10^{7} (the number and place).

Koti jya : cosine of an arc; half chord of the complementary arc.

Koti saram : radius minus bhuja-jya; Radius minus sine of the arc.

Krsna paksa (pronounced KRSH-NA PAK-SHA): the dark half of the lunar month; the period from the full moon to the new moon.

Ksetra : an enclosed figure in a plane.

Ksetra-ganitam : mathematics of (plane) figures; plane geometry.

Ksetra-phalam ; area

Kuttaka or Kuttakara : pulveriser : indeterminate equations (or Diophantine equations).

L

Laksa (pronounced LAK-SHA) : 10^{5} (the number and place).

Lipta : same as kala.

Loka : a combined name for the earth, heaven and hell.

M

Madhya : 10^{16} (the number and place); middle; center.

Mahapadma : 10^{12} (the number and place).

Mahayuga (pronounced MAHAH-YUGA) : the period consisting of the four yugas viz., krta, treta, dvapara and kali. Total duration of a mahayuga is 4,320,000 solar years. Its composition, according to the ancient Indians was as follows:-

	in solar years	
Dawn	144,00	
Golden Age (*krta yuga*)	1,440,000	
Twilight	144,000	
Total duration of *krta yuga*		1,728,000
Dawn	108,000	
Silver Age (*treta yuga*)	1,080,000	
Twilight	108,000	
Total duration of *treta yuga*		1,296,000
Dawn	72,000	
Brazen Age (*dvāpara yuga*)	720,000	
Twilight	72,000	
Total duration of *dvapara yuga*		864,000
Dawn	36,000	
Iron Age (*kali yuga*)	360,000	
Twilight	36,000	
Total duration of *kali yuga*		432,000
Total duration of a *maha yuga* :		4,320,000

Āryabhata calls a mahayuga, mentioned above, simply a yuga and speaks of (equal) quarter yugas. It is evident that Aryabhata did not accept the age-old conventions and traditional beliefs implicitly.

Malayālam (pronounced MALA-YAH-LAM); the regional language of Kerala.

Manu or Manvantara : a period of 308,448,000 solar years

14 manus + an introductory dawn = 1 kalpa,

71 mahayugas + a twilight = 1 manu

Details of the composition of a manu and kalpa, in solar years:

71 mahayugas = 71 x 4,320,000	=	306,720,000
A twilight	=	1,728,000
Duration of a manu	=	308,448,000
An introductory dawn	=	1,728,000
14 manus = 14 x 308,448,000	=	4,318,272,000
Duration of a kalpa or aeon	=	4,320,000,000

Masa : month; denotes the number 12 in the Bhūta Samkhya system

Muhurta : aspecious time or speculated time; a duration of 48 minutes.

Mūla ; the starting point of a line or arc; root; base; cause

N

Naksatra (pronounced NAK-SHA-TRA) ; star ; asterism

Netra : eye; the number 2 in the Bhūta Samkhya system.

Nikharva : 10^{11} (the number and place).

Niravasesam (pronounced NIR-AVA-SHE-SHAM): without remainder; exactly.

Niyuta : 10^5 (the number and place); same as laksa.

O

Ola : the Malayalam name for palm-leaf (see palm-leaf).

Onam : an annual festival of four days celebrated by all the people of Kerala, irrespective of their caste or creed. It occurs in August-September.

P

Pāda (pronounced PAH-DA): quadrant.

Pakazhiyam : a religious feastival.

Paksa (pronounced PAK-SHA) : one half of a lunar month from new-moon to full-moon or from full-moon to new-moon.

Pali : an ancient Indian language which was closer to the speech of the common man than was Sanskrit; its style was in general simple. It is still the religious language of the Buddhists of Ceylon, Burma and S.E. Asia. It seems to look back rather to the vedic than classical Sanskrit.

Palm leaf : the leaf of a particular variety of palm - not the coconut palm but similar to it. Its leaves are cut to the requisite size, dried and used as writing material. Documents and manuscripts to be preserved were written on this.

Panchangam : the Hindu almanac containing details regarding the five elements vāra, naksatra, tithi, yoga and karana.

Pandava : the five brothers Yudhisthira, Bhima, Arjuna, Nakula and Sahadeva (characters of the Hindu epic the Mahā Bhārata)

Pandya (pronounced PAH-NDYA): one of the three ancient dynasties in south India; the other two being the Cera and Chola.

Para : one outside the fold of some speciality like astronomy.

Parahitam : a Keralese system of astronomical calculation promulgated in 683 A.D.

Paraloka : the other world; the world of the dead.

Paralperu : an expression denoting a number in the Katapayadi system.

Parama-guru : supreme teacher.

Parardha : 10^{17} (the number and place).

Parasava (pronounced PAH-RA-SHAVA) : a community known as variyar

Parasurama (pronounced PARA-SHU-RAH-MA) : a celebrated brahmin warrior, considered as the sixth incarnation of Vishnu.

Parasuram era : an era that commenced in 1176 B.C. (not popular).

Paridhi : circumference of a circle; a boundary.

Parinaham : periphery; circumference (of a circle).

Phala : fruit; consequent; result; area.

Pratalpara : a measure of an arc of a circle. (see the table in the chapter 'Some concepts and their nomenclature')

Prayuta : 10^6 (the number and place).

Prastha (pronounced PRASH-THA); Surface.

Prastha phalam : surface area.

Purāna (Pronounced PU-RAH-NA) : well-known sacred books of the Hindus. Their number is 18. They are supposed to have been composed by the sage Vyasa and contain the whole body of Hindu mythology. Their names are (1) Brhma (2) Padma (3) Vaishnava (4) Vayavya (5) Bhagavata (6) Naradiya (7) Markandeya (8) Agneya (9) Bhavishya (10) Brhma-vaivarta (11) Laingika (12) Varaha (13) Skanda (14) Vamana (15) Kaurma (16) Matsya (17) Garuda & (18) Brahmandha.

Purna : complete : whole.

R

Radian : a unit of angle measurement; the angle subtended at the center of a circle by ar arc, whose length is equal to the radius

Randhra : hole

Rasi (Pronounced RAH-SHI): Sign of the zodiac; the arc of a circle facing an angle of 30 degrees at the center; quantity; the

number 12 in the Bhūta Samkhya system.

Rasi-chakram : the ecliptic.

Regular : applied to rectilinear geometrical figures, mean all the sides are equal.

Rk : one of the vedas (see veda)

S

Sad (pronounced SHAD) : six

Sad : good

Sad-darsana (pronounced SHAD-DARSHANA) : six doctrines of Hinduism. The six schools differ in their origin and purpose, but all are considered equally valid ways of salvation. They are divided into three pairs. They are nyaya (logic and epistomology) and vaisesika (individual characteristics); sankhya (count or enumeration) and yoga (spiritual discipline); mimamsa (enquiry) and Vedanta (the end of the vedas; also called uttara mimamsa (or sequel to the mimasa).

Sages; saints. The Hindu mythology lists seven sages; they are Angiras, Atri, Krtu, Marici, Pulaha, Pulastya and Vasistha

Sahasra : thousand (the number and place).

Saka era : one of the most wide-spread Indian system of reckoning dates; it was founded in 78 A.D.

Samadala : of equal sides

Sama-parinaha : circumference (of a circle)

Sama sadasram : regular haxagon

Samasta-jya : full-chord.

Samkhya : number.

Samkraman : the point of time when the sun, appears to enter any one of the rasis or signs of the zodiac.

Samskrita : corrected.

Savarga : multiplication; product.

Sankranti : entrance of the sun in a rasi or sign.

Sanku (pronounced SHAN-KU) : 10^{13} (the number and place).

Saram (pronounced sha-ram) : means arrow; the height or sag of a semiarc; same as bhuja-saram i.e., radius minus koti-jya; versine of the arc.

Sariram (pronounced SHA-REE-RAM) : body; solid.

Sarva-vid : one who has mastered all sastras or knowledge.

Sasi (pronounced SHA-SHI) : the moon.

Sata (pronounced SHA-TA) : hundred (the number and place).

Shaivites : devotees of Shiva (see Shiva)

Siddhantas (pronounced SIDDHAH-NTAH-S) : the Hindu astronomical works. They were alleged to be revelations of Gods (to the sages); this means that their authors have hidden their names and their times with the definite motive of making their astronomical systems and calculations acceptable without being questioned. They go by the following names: (1) Surya (2) Paitamaha (3) Vyasa (4) Vasistha (5) Atri (6) Parasara (7) Kasyapa (8) Narada (9) Gara (10) Marici (11) Manu (12) Angira (13) Lomaka (Romaka?) (14) Paulisa (15) Cyavana (16) Yavana (17) Bhrgu and (18) Saunaka,

Varaha mihira mentions only five of these Siddhantas viz.,

Paitamaha, Vasistha, Romaka, Paulisa and Surya.

Shiva : a member of the Hindu Trinity, the other two being Brhma and Vishnu.

Somayaji : one who has performed the religious function, called the soma yajna or sacrifice

Sredhi (pronounced SHRE-DHI); series ; progression.

Sredhi-ksetra (pronounced SHRE-DHI KSHE-TRA) : area of a figure represented by a series graphically.

Sukla (pronounced SHUKLA) : white; bright.

Sukla paksa (pronounced SHUKLA PAK-SHA) ; The bright half of a lunar month; the period from the new moon to the full moon.

Sulva sutra (pronounced SHUL-VA SOO-TRA) : The root meaning of the word sulv is to measure and in due course the word came to mean the rope or cord. Geometry in ancient India was for long known by the name sulva. Seven of the sulva sutras are known at present. They are known by the names : Bodhayāna, Āpasthamba, Katyāyana, Mānava, Maitrayāna, Varāha and Vadhula after the names of the sages who wrote them. Of these the Bodhayana, the Apasthamba and the Katyayana are of importance from the mathematical point of view; they contain some rules on geometrical constructions. They are supposed to have been written between 800 B.C. and 500 B.C. The sulva sutras are not formal mathematical treatises; they are only adjuncts to certain religious works.

Sunya (pronounced SHOO-NYA) : zero; void.

Sutra (pronounced SOO-TRA) : means a string; a short rule or precept; an aphorism; a formula; also a work or manual containing such aphoristic rules.

Sutra kara (pronounced SOO-TRA KAH-RA) : composer of sutras.

svam : addition; additive quantity.

T

Talpara : one sixtieth of a vili or vikala.

Taluk (pronounced TAH - LOOK) : a subdivison of a district, comprising a number of villages, placed for purposes of revenue under a native collector.

Tamil : one of the Dravidian languages, originally of ancient Pandya kingdom and of Tamil nadu at present.

Tamilakam : old name for the land of the Tamils (in south India).

Tantra : principle ; doctrine; theory; rule; method; also a class of astronomical treatises.

Tithi : lunar day; thirtieth part of a lunar month

Trairasika (pronounced TRAI-RAH-SHI-KA) : Rule of three; direct proportion.

Tribhuja : triangle.

Tri-jya : sine of three signs (or rasis) or sine of a quadrant of a circle : the radius.

Trisama : equilateral (triangle).

Tryasra (Pronounced TRY-ASH-RA) : triangle

U

Udayam : rising ; heliacal rising.

Upapatti : proof

Upanayana : a ritual for the initiation of a boy born of brahmin parents to the brahmin way of life. The boy starts wearing a sacred thread (called Yajnopaveeta in Sanskrit or poonool in Malayalam and Tamil) about his left shoulder going down to the right hip around his body as an insignia of this ritual.

Urdhva : top.

Urdva bhuja : altitude or vertical side (of a triangle).

Utkrama Jya : same as saram or versine of an arc.

V

Vaishnavite : a devotee of Vishnu (see Vishnu).

Vamana (pronounced VAH-MA-NA) : a dwarf; the fifth incarnation of

Vishnu.

Vāra (pronounced VAH-RA) : day (of the week).

Varga : square

Varga-mūla : square root.

Varsha : year

Vatsara : year

Veda : Sacred knowledge; holy learning; the scriptures of the Hindus. Originally there were only 3 vedas: the rk, the yajur, and the sama, collectively called 'the sacred triad'; but subsequently a fourth, the atharva was added, thus making them four. Hence in the Bhūta Samkhya system veda stands for the number 4.

Vedānga (pronounced VEDAH-N-GA): name of certain class of works regarded as auxiliary to the vedas and designed to aid in the correct pronounciation and interpretation of the texts. The vedangas are six in number: (1) siksha, the science of pronounciation and phonetics (2) chandas, or prosody, (3) vyakarana or grammar, (4) nirukta or etymological interpretation of the vedic texts, (5) jyotisha or astronomy and (6) kalpa or ritualistic.

Vidhi : method

Vikala : one sixtieth of an ili or kala; an arc of a circle facing an angle of one second at the center.

Vikram era : an era popular in north India, founded in 58 B.C.

Vili : same as vikala.

Vindhya hills ; a range of hills north of the region known as Deccan or south India.

Viskambha (pronounced VISH-KAM-BHA) : diameter (of a circle).

Vishnu : a member of the Hindu Trinity, the other two being Brhma and Shiva. According to legends there are ten incarnations of

Vishnu; they are (1) Matysa or fish (2) Kurma or tortoise (3) Varaha or boar (4) Nrsimha or half lion and half man (5) Vamana or dwarf (6) Parasurama (7) Rama (8) Balarama (9) Krishna and (10) Kalki.

Vistriti : breadth ; diameter (of a circle)

Vrinda : 10^9 (the number and place).

Vrtta : circle

Vrtta-phalam : area of a circle.

Vrtti : exposition.

Vrtti-kara : an expositor.

Vyāsa (pronounced VYAH-SA) : diameter of a circle.

Vyasardha (pronounced VYAH-SAH-R-DHA) : semi-diameter; radius.

Vyatipāta (pronounced VYA-TI-PAH-TA) ; when the sum of the (true) longitudes of the sun and the moon equals 180 degrees the phenomenon is called lāta-vyatipāta and when the sum of those longitudes equals 360 degrees, it is called vaidrta-vyatipāta

Vyavakalana : subtraction.

Y

Yoga : addition ; conjunction of planets.

Yuga : an era (see mahayuga).

Yugma : even (as in numbers).

Z

Zodiac : a belt of the celestial sphere extending about 8 degrees on each side of the ecliptic within which the apparent motions of the sun, moon and principal planets take place; it is divided into twelve equal parts called rasis or signs.

BIBLIOGRAPHY

1. Aaboe, A.: Episodes from the Early History of Mathematics, Random House, 1964.

2. Āgamananda Swamikal : Srimad Vishnu Puranam (tr. in Malayalam), B.V. Book Depot, Trivandrum, Kerala, 1930).

3. Archibald, R.C. : Outline of the History of Mathematics, Slaught Memorial Papers of the Mathematical Association of America, Buffalo, 1949.

4. Bag, A.K. : Mathematics in Ancient and Medieval India, Chaukhambha Orientalia, Varanasi, India, 1979.

5. Ball, W.W.R. : A Short Account of the History of Mathematics, Macmillan & Co. London 1888.

6. Bose D.M. Sen, S.N. & Subbarayappa, B.V. (Editors) : A Concise History of Science in India, Indian National Science Academy, New Delhi, 1971.

7. Boyer, C.B. : A History of Mathematics, Princeton University Press, Princeton, N.J, 1985.

8. Burgess, Rev. E (tr.) : "Surya Siddhanta", Indological Book House, New Delhi, 1977.

9. Curator, The University Manuscripts Library, Trivandrum. (Ed.) : "Laghu Bhaskariya" of Bhaskara with a commentary of Sankaranarayana, 1949.

10. Damodaran Nambiar K : "Aryabhatan" (in Malayalam) ; Mathrubhumi Weekly, pp. 15 - 16 & 41, Calicut, Kerala, October 18, 1970.

11. Datta, B & Singh, A.N. : History of Hindu Mathematics, Parts I & II Motilal Banarsidas, Lahore, India 1935 & 1938.

12. Dikshit, S.B. : Bharatiya Jyotis-sastra, Part I (tr. by Prof. R.V. Vaidya) Director-General of Observatories, New Delhi, India 1963.

13. Dvivedi, G.P. Pandit (Ed). : The Siddhanta Siromani of Bhaskaracarya, Lucknow, India, 1911.

14. Elfering, Kurt : "The area of a triangle and the volume of a pyramid as well as the area of a circle and the surface of the hemisphere in the Mathematics of Aryabhata I" (A summary of a chapter of the German translation and commentary "Die Mathematik des Aryabhata I - Text, Ubersetzung and Kommentar"), Indian Journal of History of Science, vol. XII, 1977, pp. 232-236).

15. Eves, Howard. : An Introduction to the History of Mathematics, Holt, Rinehart & Winston, New York, 1976.

16. Fleet, J.F. : "Aryabhata's system of expressing numbers", Journal of the Royal Asiatic Society of Great Britain and Ireland, 1911, pp. 109-126.

17. Ganguli S.K. : "India's Contribution to the theory of Indeterminate equations of the first degree", Journal of the Indian Mathematical Society, Vol, XIX, Part II, 1931, pp. 110-120, 129-142 & 153-168.

18. Govinda Pillai, M.K. : Manusmriti (tr. in Malayalam), S.T. Reddiar & Sons, Quilon, Kerala, 1920).

19. Gupta, R.C. : "Bhaskara I's approximation to the sine", Indian Journal of History of Science, vol. II, 1967, pp. 121-134.

20. Gupta, R.C.: "Second Order Interpolation in Indian Mathematics up to the Fifteenth Century", Indian Journal of History of Science, vol. IV, 1969, pp. 86-98.

21. Gupta, R.C. : "Fractional parts of Aryabhata's sines and certain rules found in Govindasvami's Bhasya on the Maha Bhaskariya".

Indian Journal of History of Science, vol. VI, 1971, pp. 51-59.

22. Gupta R.C. : "Nilakantha's Rectification Formula", The Mathematics Education, Vol. VI, Section B, 1971.

23. Gupta R.C. "Early Indians on Second Order Sine-differences".

Indian Journal of History of Science, Vol. VII, 1972.

24. Gupta, R.C. : "Solution of the Astronomical Triangle as found in the Tantra Samgraha (A.D. 1500)", Indian Journal of History of Science, vol. IX, 1974, pp. 86-89.

25. Gupta, R.C. : "An Indian Form of 3rd Order Taylor-series approximation of the Sine", Historia Mathematica, vol. I, 1974, pp. 282 - 289.

26. Gupta R.C. : "A Mean-Value-Type Formula for Inverse Interpolation of the Sine", The Mathematics Education, vol. X, section B, 1976, pp. 17-20.

27. Gupta, R.C. : "Paramesvara's Rule for the Circum-radius of a Cyclic Quadrilateral", Historia Mathematica vol. IV, 1977, pp. 67-74.

28. Gupta R.C. : "Some Important Mathematical Methods as Conceived in Sanskrit Language", Journal of the Department of Sanskrit, University of Delhi, vol III, 1979.

29. Inamdar, M.G. : "A Formula of Bhaskara for the chord of a circle leading to a formula for evaluating Sin A", The Mathematics Student, vol. XVIII, 1950, pp. 9 - 11.

30. Kern H. (Ed.) : Āryabhatiyam with the commentary Bhata-dipika of Paramadisvara, Leiden, Netherlands 1874).

31. Koru, P.K. : Lilavati (in Malayalam), Mathrubhumi Publishing House, Calicut, Kerala, 1954).

32. Koru, P.K. (Ed.) : Karana Paddhati of Putumana Somayaji (in Malayalam), The Astro Printing & Publishing Co., Cerp, Kerala, 1953.

33. Krishnaswami Ayyangar, A.A. : "The Hindu Sine-table", Journal of the Indian Mathematical Society vol. XV. 1923-1924.

34. Krishnaswami Ayyangar A.A. : "Remarks on Bhaskara's approximation to the Sine of an angle", The Mathematics Student, India, vol. XVIII, p.12.

35. Krishna Varier, P.U. : Sathavarsha Panjangam (Hundred Years Almanac in Malayalam), Pellissery Publications, Kottayam, Kerala, 1988. (This contains the almanac for the 105 years covering 1061 through 1165 Malabar Era, i.e. mid-August 1885 through mid-August 1990 A.D.)

36. Kunjan Pillai, S. (Ed.) : Āryabhatiyam with the Bhāsyam of Nilakantha Somasutvan, Part III - Gola padam, Trivandrum Sanskrit Series, Trivandrum, India, No. 185, 1957).

37. Kunjan Pillai, S. : The Tantra Samgraham of Nilakantha Somasutvan, Trivandrum Sanskrit Series, Trivandrum, India, No. 188, 1958.

38. Kuppanna Sastri, T.S. (Ed.) : Maha Bhaskariya of Bhaskaracarya, Madras Government Oriental Series, Madras, India, 1957.

39. Kuppanna Sastri, T.S. & Sarma, K.V. (Eds.) : Vakyakarana with commentary by Sundararaja; Kuppuswami Sastri Research Institute, Madras, India, 1962.

40. Menon, K.N. : Aryabhata, Publications Division, Ministry of Information & Broadcasting, New Delhi, India 1977.

41. Mukunda Marar, K. & Rajagopal, C.T. : "On the Hindu Quadrature of the Circle", Journal of the Bombay Branch of the Royal Asiatic Society, Bombay, India, New Series Vol. 20, 1944, pp. 65-82.

42. Naraharayya, S.N. : "Note on the Hindu table of Sines", Journal of the Indian Mathematical Society, vol. XV, Part II, 1923-1924, pp. 105-113.

43. Nayar, S.K. (Ed.) : Karana Paddhati of Putumana Somayaji (in Malayalam), Government Oriental Manuscripts Library, Madras, 1956.

44. Parameswara Aiyer, Ulloor, S. : Kerala Sahitya Charitram, (History of Kerala Literature - in Malayalam) vols. I to to III, University of Kerala, Trivandrum, Kerala, 1953-1955.

45. Parameswaran, S. : "Jyotirganitam" (Mathematics of Astronomy) in Kerala Charitram (History of Kerala - in Malayalam), Kerala History Association, Ernakulam, Kerala, vol. II, 1974).

46. Parameswaran, S. : "Demonstration of the Formula sin (x + y) = sinx. cosy + cosx . sin y in Medieval India", The Mathematics Teacher, Madras, India, vol. 23 1987, pp. 1-12.

47. Raja, K.K. : "Astronomy and Mathematics in Kerala", Adyar Library Bulletin, Madras, India, Vol. 27, 1963, pp. 118-167.

48. Rajagopal, C.T. : "A Neglected Chapter of Hindu Mathematics", Scripta Mathematica, New York, vol. XV. 1949, pp. 201-209.

49. Rajagopal, C.T. & Venkataraman, A. : " The sine and cosine power series in Hindu Mathematics", Journal of Royal Asiatic Society of Bengal - Science, Calcutta, India. vo. XV, 1949, pp. 1 -13.

50. Rajagopal, C.T. & Vedamurthi Aiyar, T.V. : "On the Hindu Proof of Gregory's series", Scripta Mathematica, New York, Vol. XVIII, 1951, pp. 65-74.

51. Rajagopal C.T. & Vedamurthi Aiyar, T.V. : "A Hindu Approximation to pi", Scripta Mathematica, New York, vol. XVIII, 1952, pp. 25-30.

52. Rajagopal, C.T. & Rangachari, M.S. : "On an untapped source of Medieval Keralese Mathematics", Archive for History of Exact Sciences, Baltimore, Maryland, Vol. 18, 1978, pp. 89-102.

53. Rajagopal, C.T. & Rangachari, M.S, : "On Medieval Kerala Mathematics", Archieve for History of Exact Sciences,

Baltimore, Maryland, vol. 35, 1986, pp. 91-99.

54. Raja Raja Varma, A.R. : Prabhandha Samgraham (A collection of Essays in Malayalam), Kamalalaya Book Depot, Trivandrum, Kerala, 1937).

55. Raja Raja Varma, Vatakkumkur : Kerala Samskrita Sahitya Charitram (History of Sanskrit Literature in Kerala, in Malayalam), Kamalalaya Book Depot, Trivandrum, Kerala, Vol I, 1937 - 1938.

56. Rama Varma, H.H. : "The Date and Authorship of Karana Paddhati" in Gomati Special Issue, Trichur, Kerala, 1945.

57. Rama Varma (Maru) Tampuran : "Vrittaparidhiyilute" (On the Circumference of a Circle, in Malayalam), Mangalodayam Monthly, Trichur, Kerala, 1944).

58. Rama Varma (Maru) Tampuran & Akhileswara Aiyar, A.R. (Eds.): Yukti Bhasa Part I (in Malayalam), Mangalodayam Ltd., Trichur, Kerala, 1948.

59. Sambasiva Sastri, K. (Ed.) : Aryabhatiyam - Ganita padam, University of Kerala, Trivandrum, Kerala, 1977.

60. Sambasiva Sastri, K. (Ed.) : Aryabhatiyam - Kalakriya padam, University of Kerala, Trivandrum, Kerala, 1977.

61. Sankara Menon, K. (Ed.) Aryabhatiyam - Gitika padam (in Malayalam), Government of Travancore, Trivandrum, Kerala, 1927.

62. Sankara Menon, K. (Ed.) : Jataka Paddhati of Vatasseri Paramesvaran Namputiri, The Sri Vanci Setu Lakshmi Bhai Series, Nos. 2, Trivandrum, Kerala, 1926.

63. Sankara Varma of Katattanat : Sadratnamala, with auto-commentary (in Malayalam), Janaranjini Press, Nadapuram, Kerala, 1898.

64. Sankunni, Kottarathil : Aitihyamala (in Malayalam), Kottarathil Sankunni Memorial Committee, Kottayam, Kerala, 1989.

65. Saraswati Amma, T.A. : "The Development of Mathematical Series in India", Bulletin of the National Institute of Sciences, New Delhi, India, vol. 21, 1968.

66. Saraswati Amma, T.A. : "Development of Mathematical Ideas in India", Indian Journal of History of Science, New Delhi, India, vol. IV, 1969, pp. 59-78.

67. Saraswati Amma, T.A. : Geometry in Ancient and Medieval India, Motilal Banarsidas, Delhi, India, 1979.

68. Sarma, K.V. : "Parahitattinte Mulagrantham" (Original text of Parahita - in Malayalam), Mathrubhumi weekly, Calicut, Kerala, October 10, 1954.

69. Sarma, K.V. : "Putumana Comatiriyute Avijnata Kritikal" (Unknown works of Putumana Somayaji - in Malayalam), Mathrubhumi weekly, Calicut, Kerala, January 29, 1956).

70. Sarma, K.V. : "Putumana Comatiriyute Kālam" (Date of Putumana Somayaji - in Malayalam), Mathrubhumi weekly, Calicut, Kerala, February 5, 1956).

71. Sarma, K.V. : "Anpattiyancu kollatte tapassu" (Fifty five years of penance - in Malayalam), Mathrubhumi weekly, Calicut, Kerala, October 7, 1956.

72. Sarma, K.V. : "Oru Jyotisa Granthavari" (A Series of books on Astronomy - in Malayalam), Mathrubhumi weekly, Calicut, Kerala, May 17, 1957.

73. Sarma, K.V. : "Drg-ganita of Paramesvara" (in Malayalam), Mathrubhumi weekly, Calicut, Kerala, August 28, 1960.

74. Sarma, K.V. : A History of the Kerala School of Hindu Astronomy, Vishveshvaranand Institute, Hoshiarpur, Punjab, India, 1972.

75. Sarma, K.V. (Ed.) : Grahacaranibhandhana of Haridatta,

Kuppuswami Sastri Research Institute, Madras, India, 1954.

76. Sarma, K.V. (Ed.) : "Siddhanta darpanam of Nilakantha Somayaji", Adyar Library Bulletin, Madras, India, vol. XIX, 1955.

77. Sarma, K.V. (Ed.) : Venvaroha of Madhavan of Sangamagramam, with the commentary of Acyuta Pisaroti, The Sanskrit College Committee, Tripunithura, Kerala, 1956.

78. Sarma, K.V. (Ed.) : Goladipika of Paramesvara, The Adyar Library & Research Centre, Madras, India, 1956-1957.

79. Sarma, K.V. (Ed.) : Lilavati of Bhaskaracarya with the commentary Kriyakramakari, Vishveshvaranand Institute, Hoshiarpur, Punjab, India, 1957.

80. Sarma, K.V. (Ed.) : "Grahanastaka of Paramesvara", The Journal of Oriental Research, Madras, India, Vol. XXVIII, 1958-59.

81. Sarma, K.V. (Ed.) : Drg-ganita of Paramesvara, V.V.R. Institute, Hoshiarpur, Punjab, India, 1963.

82. Sarma, K.V. (Ed.) : Grahana-mandana of Paramesvara, V.V.R. Institute, Hoshiarpur, Punjab, India, 1965.

83. Sarma, K.V. (Ed.) : Grahana-nyaya-dipika of Paramesvara, V.V.R. Institute, Hoshiarpur, Punjab, India, 1966.

84. Sarma, K.V.(Ed.) : Golasāra of Nilakantha Somayaji, Vishveshvaranand Institute, Hoshiarpur, Punjab, India, 1970.

85. Sarma, K.V. (Ed.) : Sphutacandrapti of Mādhavan of Sangamagramam, Vishveshvaranand Institute, Hoshiarpur, Punjab, India, 1973.

86. Sarma, K.V. (Ed.) : Āryabhatiya with the commentary of Surya Deva Yajvan, Indian National Science Academy, New Delhi, India, 1976.

87. Sarma, K.V. (Ed.) : Candracchaya-ganitam of Nilakantha Somayaji, V.V.B.I.S. & I.S., Hoshiarpur, Punjab, India, 1976.

88. Sarma, K.V. (Ed.) : Siddhanta Darpanam of Nilakantha, V.V.B.I.S. & I.S., Hoshiarpur, Punjab, India, 1977.

89. Sarma, K.V. (Ed.) : Tantra Sangraha of Nilakantha, V.V.B.I.S. & I.S., Hoshiarpur, Punjab, India, 1977.

90. Sarma, K.V. (Ed.) : Jyotirmimamsa of Nilakantha, V.V.B.I.S. & I.S., Hoshiarpur, Punjab, India, 1977.

91. Scott, J.F. : A History of Mathematics, Taylor & Francis, London, 1960.

92. Sen, Gertrude Emerson : The Pageant of India's History, Volume I, Longmas, Green and Co., New York, 1948.

93. Shanks, Daniel & Wrench Jr., John W. : "Calculation of pi to 100,000 decimals", Mathematics of Computation, vol. XVI, No. 77, 1962.

94. Shukla, K.S. (Ed.) : Maha Bhaskariya, Department of Mathematics & Astronomy, Lucknow University, Lucknow, India, 1960.

95. Shukla, K.S. (Ed.) : Maha Bhaskariya, Department of Mathematics & Astronomy, Lucknow University, Lucknow, India, 1963.

96. Shukla, K.S. : Āryabhata, Indian National Science Academy, New Delhi, India, 1976.

97. Shukla, K.S. (Ed.) : Āryabhatiya with commentary by Bhaskara I and Somesvara, Indian National Science Academy, New Delhi, India, 1976.

98. Shukla, K.S. & Sarma, K.V. : Āryabhatiya with translation and notes, Indian National Science Academi, New Delhi, India, 1976.

99. Singh, A.N. : "On the Indian Method of root extraction", Bulletin of Calcutta Mathematical Society, Calcutta, India, Vol. XVIII, 1927.

100. Smith, D.E. : History of Mathematics, Vols. I & II, Dover Pub.

Inc., New York, 1953.

101. Somayaji, D.A. : A Critical Study of the Ancient Hindu Astronomy, Karanataka University, Darwar, India, 1971.

102. Srinivasiengar, C.N. : The History of Ancient Indian Mathematics, The World Press, Calcutta, India, 1967.

103. Sthanunatha Pillai & Ramalingam Pillai, T. : "Āryabhatacarya & Jyotisam" (Aryabhata and Astronomy - in Malayalam), Bhasa Posini, Kerala, 1915-1918.

104. Struika, D.J. : A Concise History of Mathematics, Dover Pub. Inc., New York, 1948.

105. Taliaferro, R.C. (translator) : Almagest of Ptolemy in Hutchins, R.M. (Editor) : Great Books of the Western World, Vol. 16, University of Chicago, Chicago, 1952.

106. Thibaut, G. & Dvivedi, S. (Eds.) : Panchasiddhantika of Varahamihira, Chaukamba Publications, Varanasi, India, 1968.

107. Venkitaraman, A. : "Some Interesting Proofs from Yukto Bhasa", The Mathematics Student, India, vol. XVI, 1948, pp. 1-7.

108. Weil, Andre : Number Theory, Birkhauser Boston Inc., Boston, 1984.

109. Whish, C.M. : "On the Hindu Quadrature of the Circle and infinite Series of the proportion of the circumference to the diameter exhibited in the four Sastras, the Tantra Sangraham, Yucti Bhāsa, Carana Paddhati, and Sadratnamala", Transactions of the Royal Asiatic Society of Great Britain & Ireland, vol. III, 1835, pp. 509 - 523.